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► **To cite this version:**

Benjamin Melinand. Rigid lid limit in shallow water over a flat bottom. *Studies in Applied Mathematics*, 2024, 153 (4), pp.e12773. 10.1111/sapm.12773 . hal-04950473

HAL Id: hal-04950473

<https://univ-paris-dauphine.hal.science/hal-04950473v1>

Submitted on 16 Feb 2025

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Rigid lid limit in shallow water over a flat bottom

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Funding information

ANR project CRISIS, Grant/Award Number: ANR-20-CE40-0020-01

Abstract

We perform the so-called rigid lid limit on different shallow water models such as the abcd Boussinesq systems or the Green–Naghdi equations. To do so, we consider an appropriate nondimensionalization of these models where two small parameters are involved: the shallowness parameter μ and a parameter ϵ which can be interpreted as a Froude number. When the parameter ϵ tends to zero, the surface deformation formally goes to the rest state, hence the name rigid lid limit. We carefully study this limit for different topologies. We also provide rates of convergence with respect to ϵ and careful attention is given to the dependence on the shallowness parameter μ .

KEYWORDS

dispersive estimates, singular limit, shallow water, water waves

1 | INTRODUCTION

We consider shallow water asymptotic models of the water wave equations that can be written under the general form

$$\epsilon \partial_t \mathbf{U} + \mathcal{L}_\mu \mathbf{U} + \epsilon Q_\mu(\mathbf{U}) = 0.$$

Here $t \in \mathbb{R}^+$, $x \in \mathbb{R}$ or \mathbb{R}^2 , $\mathbf{U} = (\zeta, \mathbf{V}) \in \mathbb{R}^2$ or \mathbb{R}^3 , $1 + \epsilon \zeta$ is the nondimensionalized water depth, \mathbf{V} is the nondimensionalized horizontal velocity of the fluid, \mathcal{L}_μ is a linear operator that tends to the wave operator as $\mu \rightarrow 0$ and Q_μ a nonlinear operator. The nondimensionalized parame-

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ter μ measures the shallowness of the flow. The nondimensionalized parameter ϵ compares the amplitude of the water waves to the water depth and can also be seen as a comparison between the typical horizontal velocity of the fluid and the typical velocity of the water waves. Note that the nondimensionalized parameter ϵ appears in front of ∂_t since the characteristic time used to nondimensionalize the time variable is the typical timescale of the fluid (and not the one of water waves). We assume in the following that ϵ and μ both belong to $(0,1]$.

The goal of this paper is to perform the limit $\epsilon \rightarrow 0$ and understand how the parameter μ interferes with this convergence. In our framework, the velocity component \mathbf{V} tends to a solution of the incompressible Euler equations whereas the free surface ζ tends to 0 thus the terminology rigid lid limit. Actually, the rigid lid approximation is a common assumption in the oceanographical literature and can be interpreted as a low Froude number assumption (ocean currents travel slower than water waves). We refer for instance to Refs. 1–3 where existence and stability of nearshore shear waves that are not gravity waves were discussed, and such an assumption is used. In the same direction, Camassa et al.^{4,5} derive from the free surface Euler equations two asymptotic models called nowadays the lake equations and the great lake equations where again a rigid lid approximation is considered. In our setting, the lake equations are obtained by neglecting terms of order $\mathcal{O}(\mu)$ and then performing the limit $\epsilon \rightarrow 0$, whereas the great lake equations are derived by neglecting terms of order $\mathcal{O}(\mu^2)$ and then performing the limit $\epsilon \rightarrow 0$. A full justification of the lake equations was obtained in Ref. 6 (see also Ref. 7). We emphasize that all the previously cited works consider a nonflat bathymetry. That is not the case in this paper where the seabed is assumed to be flat. Finally, we also refer to Ref. 8, where a first study of the rigid lid limit on the full irrotational water wave equations was performed. The proof is, however, based on weighted dispersive estimates that are not well suited for the local well-posedness on large time, and the rates of convergence obtained are not optimal. Using the strategy provided in this paper, one can improve⁸ by obtaining rates of convergence as those established, for instance, in Section 5 for the irrotational Green–Naghdi equations.

We carefully study if the convergence is strong, meaning in $L^\infty(0, T; L^2(\mathbb{R}^n))$ for some $T > 0$ independent of ϵ . If not, we provide the default of strong convergence, meaning a corrector $\hat{\mathbf{U}}_\epsilon$ such that $\mathbf{U} - \hat{\mathbf{U}}_\epsilon$ strongly converges. We also get convergence in $L^q(0, T; L^\infty(\mathbb{R}^n))$ for some $q \geq 2$ (weak convergence) and in $L^2(0, T; L^2_{loc}(\mathbb{R}^n))$ (convergence of the local energy).^{*} All our convergence results are given with rates of convergence in terms of ϵ that crucially depend on the shallowness parameter μ and the function space we consider. Our strategy is the following. We assume bounds on appropriate H^s -norms of \mathbf{U} that are uniform with respect to ϵ on an existence time,[†] that is independent of ϵ . Using Duhamel’s formulation, one can then use Strichartz estimates to get $L^q_t L^r_x$ bounds and Morawetz-type estimates to obtain $L^2_t L^2_{loc}$ controls.

Our problem shares some similarities with the incompressible limit (low Mach number limit) of the compressible Euler equations defined in \mathbb{R}^n :

$$\begin{cases} \epsilon \partial_t c + \epsilon \mathbf{V} \cdot \nabla c + \frac{\gamma - 1}{2} (1 + \epsilon c) \nabla \cdot \mathbf{V} = 0, \\ \epsilon \partial_t \mathbf{V} + \epsilon (\mathbf{V} \cdot \nabla) \mathbf{V} + \frac{\gamma - 1}{2} (1 + \epsilon c) \nabla c = 0. \end{cases}$$

^{*} (Note that this kind of convergence was also obtained for the rigid lid limit of the shallow water equations over a nonflat bottom in Ref. 6 (without any rate of convergence); see Theorem 2.5 and Subsection 4.2.2 inside.)

[†] (Such bounds and existence time can be obtained from energy estimates when one prove the local well-posedness; see, for instance, Subsections 2.1 or 3.1 below.)

In that case, $1 + \epsilon c$ is the rescaled speed of the sound, \mathbf{V} is the velocity of the fluid, $\gamma > 1$ is the adiabatic exponent, and ϵ is the Mach number. The limit $\epsilon \rightarrow 0$ was studied by several authors and we refer, for instance, to Refs. 9–14. It is now well understood that the acoustic component of (c, \mathbf{V}) is of dispersive type and weakly converges to 0 since its propagation speed is of size $\frac{1}{\epsilon}$ and, when the initial datum only contains incompressible terms, the convergence is strong. Such a phenomenon also appears for the rigid lid limit.

Finally, we mention several works^{15–18} where dispersive estimates similar to ours are used to study the long-time existence problem on Boussinesq and Boussinesq-type systems.

We organize the paper as follows. Sections 2 and 3 are devoted to the study of the rigid lid limit on the classical Boussinesq system (also called the Amick–Schonbek system in the literature), respectively, in one and two dimensions. We explain in detail our strategy. In Section 4, we consider other Boussinesq systems. Finally, in Section 5, we perform the rigid lid limit on the Green–Naghdi equations. Appendix A recalls some basic facts about Littlewood–Paley decomposition, and Appendix B gathers different useful Fourier multiplier estimates. Finally, we provide, in Appendix C, general dispersive estimates on linear dispersive equations with radial nonhomogeneous phases.

Notations

- If f is a Schwartz class function defined on \mathbb{R}^n , we define $\mathcal{F}f$ or \hat{f} as the Fourier transform of f by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx.$$

- If m is a smooth function that is at most polynomial at infinity, we define the Fourier multiplier $m(D)$ as, for any Schwartz class function f ,

$$m(D)f = \mathcal{F}^{-1}(m(\xi)\hat{f}(\xi)).$$

Note that the Laplace operator verifies $\Delta = -|D|^2$.

- If f is a function defined on \mathbb{R}^n , we denote by $\|f\|_{L^p}$ the $L^p(\mathbb{R}^n)$ norm of f .
- If $p \in [1, \infty]$, we denote $p' = \frac{p}{p-1}$.
- If $T > 0$ and $G : (t, x) \in [0, T] \times \mathbb{R}^n \rightarrow G(t, x) \in \mathbb{R}$, the norm $\|G\|_{L_t^q L_x^r}$ corresponds to the norm of the space $L^q((0, T); L^r(\mathbb{R}^n))$.
- If $S : E \rightarrow F$ is a linear bounded operator with E, F two Banach spaces, we denote by S^* its adjoint.

2 | THE CLASSICAL 1D BOUSSINESQ SYSTEM

In the one-dimensional case, the classical Boussinesq system reads as

$$\begin{cases} \epsilon \partial_t \zeta + \partial_x ([1 + \epsilon \zeta] V) = 0, \\ \epsilon \left(1 - \frac{\mu}{3} \partial_x^2\right) \partial_t V + \partial_x \zeta + \epsilon V \partial_x V = 0. \end{cases} \quad (1)$$

Denoting by $\mathbf{U} = (\zeta, V)^T \in \mathbb{R}^2$, we get the following system:

$$\epsilon \partial_t \mathbf{U} + A(\partial_x) \mathbf{U} = \epsilon F(\mathbf{U}) \tag{2}$$

where

$$A(\partial_x) = \begin{pmatrix} 0 & \partial_x \\ \left(1 - \frac{\mu}{3} \partial_x^2\right)^{-1} \partial_x & 0 \end{pmatrix} \text{ and } F(\mathbf{U}) = - \begin{pmatrix} \partial_x(\zeta V) \\ \frac{1}{2} \left(1 - \frac{\mu}{3} \partial_x^2\right)^{-1} \partial_x(V^2) \end{pmatrix}.$$

Note that

$$\exp(tA(\partial_x)) = \begin{pmatrix} \cos\left(t \frac{D}{\sqrt{1 + \frac{\mu}{3} D^2}}\right) & i \sqrt{1 + \frac{\mu}{3} D^2} \sin\left(t \frac{D}{\sqrt{1 + \frac{\mu}{3} D^2}}\right) \\ \frac{i}{\sqrt{1 + \frac{\mu}{3} D^2}} \sin\left(t \frac{D}{\sqrt{1 + \frac{\mu}{3} D^2}}\right) & \cos\left(t \frac{D}{\sqrt{1 + \frac{\mu}{3} D^2}}\right) \end{pmatrix}. \tag{3}$$

We state the main result of this section.

Theorem 1. *Let $M > 0, T > 0, \epsilon \in (0, 1]$ and $\mu \in (0, 1]$. Let $(\zeta, V) \in C([0, T]; (H^3 \times H^3)(\mathbb{R}))$ a solution of (1) with initial datum (ζ_0, V_0) such that*

$$\|(\zeta, V)\|_{L^\infty(0, T; H^3(\mathbb{R}) \times H^3(\mathbb{R}))} \leq M.$$

There exists a constant $C > 0$ independent of M, T, ϵ and μ such that for any $q, r \geq 2$ with $\frac{1}{q} + \frac{1}{2r} = \frac{1}{4}$ and any $\tilde{q}, \tilde{r} \geq 2$ with $\frac{1}{\tilde{q}} + \frac{1}{3\tilde{r}} = \frac{1}{6}$

$$\begin{aligned} \left\| \begin{pmatrix} \zeta \\ V \end{pmatrix} - e^{-\frac{t}{\epsilon} A(\partial_x)} \begin{pmatrix} \zeta_0 \\ V_0 \end{pmatrix} \right\|_{L_t^q(0, T; L_x^r(\mathbb{R}))} &\leq \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{4} + \frac{1}{q}} M^2 T^{\frac{3}{4}} C, \\ \left\| \begin{pmatrix} \zeta \\ V \end{pmatrix} - e^{-\frac{t}{\epsilon} A(\partial_x)} \begin{pmatrix} \zeta_0 \\ V_0 \end{pmatrix} \right\|_{L_t^{\tilde{q}}(0, T; L_x^{\tilde{r}}(\mathbb{R}))} &\leq \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{6} + \frac{1}{\tilde{q}}} M^2 T^{\frac{5}{6}} C, \\ \left\| \begin{pmatrix} \zeta \\ V \end{pmatrix} \right\|_{L_t^{\tilde{q}}(0, T; L_x^{\tilde{r}}(\mathbb{R}))} &\leq \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{\tilde{q}}} \left(M + M^2 T^{\frac{5}{6}} \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{6}} \right) C, \\ \sup_{x_0 \in \mathbb{R}} \left\| e^{-(x-x_0)^2} \begin{pmatrix} \zeta \\ V \end{pmatrix} \right\|_{L_t^2(0, T; L_x^2(\mathbb{R}))} &\leq \epsilon^{\frac{1}{2}} (M + M^2 T) C. \end{aligned}$$

Remark 1. Several remarks are in order. We first note that (ζ, V) weakly converges to 0 as $\epsilon \rightarrow 0$. However (ζ, V) does not strongly converge to 0, meaning in $L^\infty(0, T; L^2(\mathbb{R}))$, except if $(\zeta_0, V_0) = (0, 0)$. The first two estimates provide the default of strong convergence. The first three

estimates are not uniform with respect to $\mu \rightarrow 0$. The fourth estimate provides the convergence to 0 in $L^2_t(0, T; H^s_{loc}(\mathbb{R}))$ for any $s \in [0, 3)$ and is uniform with respect to $\mu \rightarrow 0$. That corresponds to the decay to 0 of the local energy.

Remark 2. Note that if $\epsilon \sim \mu$ as in Refs. 19, 20 or when $\mu = \mathcal{O}(\epsilon)$, the first two estimates do not provide a convergence result as $\epsilon \rightarrow 0$. It is known in that case that nonlinear terms must be taken into account and that asymptotic models like a system of decoupling KdV equations or a system of decoupling BBM equations become relevant.

A proof of such a result can be adapted from Ref. 20 or Ref. [21, Section 7.1].

2.1 | Local existence

In this section, we provide a local well-posedness result of (1) on an existence time independent of $\epsilon, \mu \in (0, 1]$. We introduce the functional space

$$X^3_\mu(\mathbb{R}) := \{(\zeta, V) \in (H^3 \times H^4)(\mathbb{R}), \|(\zeta, V)\|_{X^3_\mu} := \|\zeta\|_{H^3} + \|V\|_{H^3} + \sqrt{\mu}\|\partial_x V\|_{H^3} < \infty\}.$$

Proposition 1. *Let $h_0 > 0, A > 0, \epsilon \in (0, 1]$ and $\mu \in (0, 1]$. Let $(\zeta_0, V_0) \in X^3_\mu(\mathbb{R})$ with $1 + \epsilon\zeta_0 \geq h_0$ and $\|(\zeta_0, V_0)\|_{X^3_\mu} \leq A$. There exists a time $T > 0$ that only depends on A and h_0 and a unique solution $(\zeta, V) \in C([0, T]; X^3_\mu(\mathbb{R}))$ of (1) with initial datum (ζ_0, V_0) . Furthermore, there exists a constant $M > 0$ depending only on A and h_0 such that*

$$\|(\zeta, V)\|_{L^\infty(0, T; X^3_\mu)} \leq M.$$

The proof is similar to the proofs of Ref. [22, Proposition 2.15] or Ref. [23, Theorem 1]. The key point is that System (1) under the assumption that $(1 + \epsilon\zeta) \geq \frac{h_0}{2}$ has a symmetrizer

$$S(U) = \begin{pmatrix} 1 & 0 \\ 0 & (1 + \epsilon\zeta) - \frac{\mu}{3}\partial_x((1 + \epsilon\zeta)\partial_x \cdot) \end{pmatrix}$$

and corresponding energies

$$\mathcal{E}^k(U) = (S(U)\partial_x^k U, \partial_x^k U)_2$$

that allow to control the $X^3_\mu(\mathbb{R})$ -norm.

2.2 | Dispersive estimates

The phase $g := r \mapsto \frac{r}{\sqrt{1 + \frac{r^2}{3}}}$ is smooth, and

$$g' > 0, g'' < 0 \text{ on } \mathbb{R}_*^+, g'(r) - 1 \underset{r \sim 0}{\sim} -\frac{r^2}{2}, g''(r) \underset{r \sim 0}{\sim} -r, g'(r) \underset{r \sim +\infty}{\sim} 3^{\frac{3}{2}}r^{-3}, g''(r) \underset{r \sim +\infty}{\sim} 3^{\frac{5}{2}}r^{-4}. \quad (4)$$

Thanks to Appendix C, one can prove several dispersive estimates. In the following, χ is a smooth compactly supported function that is equal to 1 near 0.

First, using Lemma C1,[‡] there exists a constant $C > 0$ such that for any Schwartz class function f , any $t \neq 0$, any $\mu \in (0, 1]$ and any $\epsilon \in (0, 1]$

$$\begin{aligned} \left\| e^{\pm \frac{t}{\epsilon} \frac{\partial_x}{\sqrt{1+\frac{\mu}{3}D^2}}} \chi(\sqrt{\mu}|D|)\text{sgn}(D)|D|^{\frac{1}{2}} f \right\|_{L_x^\infty} &\leq \frac{C}{\mu^{\frac{1}{2}}} \left(\frac{\epsilon}{|t|} \right)^{\frac{1}{2}} \left\| \chi(\sqrt{\mu}|D|)f \right\|_{L^1}, \\ \left\| e^{\pm \frac{t}{\epsilon} \frac{\partial_x}{\sqrt{1+\frac{\mu}{3}D^2}}} (1 - \chi(\sqrt{\mu}|D|))f \right\|_{L_x^\infty} &\leq \frac{C}{\mu^{\frac{1}{2}}} \left(\frac{\epsilon}{|t|} \right)^{\frac{1}{2}} \left\| (1 - \chi(\sqrt{\mu}|D|))(\sqrt{\mu}|D|)^2 f \right\|_{L^1}. \end{aligned}$$

Then corresponding Strichartz estimates can be obtained: for any $(q, r) \in \{(4, \infty), (\infty, 2)\}$ and any functions f and G are smooth enough

$$\begin{aligned} \left\| e^{\pm \frac{t}{\epsilon} \frac{\partial_x}{\sqrt{1+\frac{\mu}{3}D^2}}} \partial_x f \right\|_{L_t^q L_x^r} &\lesssim \left(\frac{\epsilon}{\mu} \right)^{\frac{1}{2} \left(\frac{1}{2} - \frac{1}{r} \right)} \left\| |D|^{\frac{3}{4} + \frac{1}{2r}} \chi(\sqrt{\mu}|D|)f \right\|_{L^2} \\ &\quad + \left(\frac{\epsilon}{\mu} \right)^{\frac{1}{2} \left(\frac{1}{2} - \frac{1}{r} \right)} \left\| (1 - \chi(\sqrt{\mu}|D|))(\sqrt{\mu}|D|)^{2 \left(\frac{1}{2} - \frac{1}{r} \right)} \partial_x f \right\|_{L^2} \\ \left\| \int_0^t e^{\pm \frac{(t-s)}{\epsilon} \frac{\partial_x}{\sqrt{1+\frac{\mu}{3}D^2}}} \partial_x G(s) ds \right\|_{L_t^q L_x^r} &\lesssim \left(\frac{\epsilon}{\mu} \right)^{\frac{1}{2} \left(1 - \frac{1}{r} \right)} \left\| |D|^{\frac{1}{2} + \frac{1}{2r}} \chi(\sqrt{\mu}|D|)G \right\|_{L_t^{\frac{4}{3}} L_x^1} \\ &\quad + \left(\frac{\epsilon}{\mu} \right)^{\frac{1}{2} \left(1 - \frac{1}{r} \right)} \left\| (1 - \chi(\sqrt{\mu}|D|))(\sqrt{\mu}|D|)^{2 \left(1 - \frac{1}{r} \right)} \partial_x G \right\|_{L_t^{\frac{4}{3}} L_x^1}. \end{aligned} \tag{5}$$

One can prove such bounds by applying a T^*T argument[§] on the following operators that are both defined from $L_t^{q'}(\mathbb{R}, L_x^r(\mathbb{R}))$ to $L^2(\mathbb{R})$

$$\begin{aligned} T_{LF}(F) &= \int_{\mathbb{R}} e^{\mp \frac{s}{\epsilon} \frac{\partial_x}{\sqrt{1+\frac{\mu}{3}D^2}}} \chi(\sqrt{\mu}|D|) \frac{\partial_x}{|D|^{\frac{3}{4} + \frac{1}{2r}}} F(s) ds, \\ T_{HF}(F) &= \int_{\mathbb{R}} e^{\mp \frac{s}{\epsilon} \frac{\partial_x}{\sqrt{1+\frac{\mu}{3}D^2}}} \frac{1 - \chi(\sqrt{\mu}|D|)}{(\sqrt{\mu}|D|)^{1 - \frac{2}{r}}} F(s) ds. \end{aligned}$$

[‡] with $\beta = 1, s = \frac{1}{2}, \alpha = -4, l = 2$.

[§] in the spirit of Ref. [24, Lemma 2.1].

In a similar way, one can also get from Lemma C1[¶] together with Lemma B1 that for any functions f and F are smooth enough

$$\left\| e^{\pm \frac{t}{\epsilon} \frac{\partial_x}{\sqrt{1 + \frac{\mu}{3} D^2}}} f \right\|_{L^\infty} \lesssim \frac{C}{\mu^{\frac{1}{3}}} \left(\frac{\epsilon}{|t|} \right)^{\frac{1}{3}} \|(1 + \mu D^2)^{\frac{5}{6}} f\|_{L^1}$$

and corresponding Strichartz estimates can be obtained from a T^*T argument: for any $(\tilde{q}, \tilde{r}) \in \{(6, \infty), (\infty, 2)\}$

$$\left\| e^{\pm \frac{t}{\epsilon} \frac{\partial_x}{\sqrt{1 + \frac{\mu}{3} D^2}}} f \right\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \lesssim \left(\frac{\epsilon}{\mu} \right)^{\frac{1}{3} \left(\frac{1}{2} - \frac{1}{\tilde{r}} \right)} \left\| (1 + \mu D^2)^{\frac{5}{6} \left(\frac{1}{2} - \frac{1}{\tilde{r}} \right)} f \right\|_{L^2} \tag{6}$$

$$\left\| \int_0^t e^{\pm \frac{(t-s)}{\epsilon} \frac{\partial_x}{\sqrt{1 + \frac{\mu}{3} D^2}}} F(s) ds \right\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \lesssim \left(\frac{\epsilon}{\mu} \right)^{\frac{1}{3} \left(1 - \frac{1}{\tilde{r}} \right)} \left\| (1 + \mu D^2)^{\frac{5}{6} \left(1 - \frac{1}{\tilde{r}} \right)} F \right\|_{L_t^{\frac{6}{5}} L_x^1} \tag{7}$$

2.3 | Proof of Theorem 1

From Duhamel’s principle on (2)

$$\begin{pmatrix} \zeta \\ V \end{pmatrix} (t) - e^{-\frac{t}{\epsilon} A(\partial_x)} \begin{pmatrix} \zeta_0 \\ V_0 \end{pmatrix} = \int_0^t \exp\left(\frac{\tau-t}{\epsilon} A(\partial_x)\right) F \left(\begin{pmatrix} \zeta \\ V \end{pmatrix} (\tau) \right) d\tau.$$

Then from (3) and since F is a derivative we can use (5) so that for any $(q, r) \in \{(4, \infty), (\infty, 2)\}$

$$\left\| \int_0^t \exp\left(\frac{\tau-t}{\epsilon} A(\partial_x)\right) F \left(\begin{pmatrix} \zeta \\ V \end{pmatrix} (\tau) \right) d\tau \right\|_{L_t^q(0, T; L_x^r(\mathbb{R}))} \lesssim \left(\frac{\epsilon}{\mu} \right)^{\frac{1}{4} + \frac{1}{q}} (A_{LF} + A_{HF})$$

where

$$\begin{aligned} A_{LF} = & \left\| |D|^{\frac{1}{2} + \frac{1}{2r}} (\zeta V) \right\|_{L_t^{\frac{4}{3}}(0, T; L_x^1)} + \left\| \frac{|D|^{\frac{1}{2} + \frac{1}{2r}}}{\sqrt{1 + \frac{\mu}{3} |D|^2}} (V^2) \right\|_{L_t^{\frac{4}{3}}(0, T; L_x^1)} \\ & + \left\| \frac{|D|^{\frac{1}{2} + \frac{1}{2r}}}{\sqrt{1 + \frac{\mu}{3} |D|^2}} (\zeta V) \right\|_{L_t^{\frac{4}{3}}(0, T; L_x^1)} + \left\| \frac{|D|^{\frac{1}{2} + \frac{1}{2r}}}{1 + \frac{\mu}{3} |D|^2} (V^2) \right\|_{L_t^{\frac{4}{3}}(0, T; L_x^1)} \end{aligned}$$

[¶] with $\beta = 1, s = 0, \alpha = -4, l = 3$.

and

$$\begin{aligned}
 A_{HF} = & \left\| (\sqrt{\mu}|D|)^{2(1-\frac{1}{r})} \partial_x(\zeta V) \right\|_{L_t^{\frac{4}{3}}(0,T;L_x^1)} + \left\| \frac{(\sqrt{\mu}|D|)^{2(1-\frac{1}{r})}}{\sqrt{1+\frac{\mu}{3}|D|^2}} \partial_x(V^2) \right\|_{L_t^{\frac{4}{3}}(0,T;L_x^1)} \\
 & + \left\| \frac{(\sqrt{\mu}|D|)^{2(1-\frac{1}{r})}}{\sqrt{1+\frac{\mu}{3}|D|^2}} \partial_x(\zeta V) \right\|_{L_t^{\frac{4}{3}}(0,T;L_x^1)} + \left\| \frac{(\sqrt{\mu}|D|)^{2(1-\frac{1}{r})}}{1+\frac{\mu}{3}|D|^2} \partial_x(V^2) \right\|_{L_t^{\frac{4}{3}}(0,T;L_x^1)}.
 \end{aligned}$$

Lemmas B1 and B4 provide

$$A_{LF} + A_{HF} \lesssim \| \zeta \|_{H_x^3} \| V \|_{H_x^3} \| \cdot \|_{L_t^{\frac{4}{3}}(0,T)} + \| \| V \|^2 \|_{H_x^3} \| \cdot \|_{L_t^{\frac{4}{3}}(0,T)} \lesssim T^{\frac{3}{4}} M^2.$$

We then get the first bound when $(q, r) \in \{(4, \infty), (\infty, 2)\}$ and the other cases follow from Hölder’s inequality. The second bound follows the same way using instead (6) and (7). Note that since the initial datum is not necessarily a derivative one cannot use the homogeneous version of (5), hence a weaker rate of convergence in that case. Finally, the third bound follows from Morawetz-type estimates established in Proposition 4.

3 | THE CLASSICAL 2D BOUSSINESQ SYSTEM

The two-dimensional classical Boussinesq system reads as

$$\begin{cases} \epsilon \partial_t \zeta + \nabla \cdot ([1 + \epsilon \zeta] \mathbf{V}) = 0, \\ \epsilon \left(1 - \frac{\mu}{3} \nabla \nabla \cdot \right) \partial_t \mathbf{V} + \nabla \zeta + \epsilon (\mathbf{V} \cdot \nabla) \mathbf{V} = 0. \end{cases} \tag{8}$$

Taking the divergence of the second equation and denoting by $\mathbf{U} := (\zeta, \nabla \cdot \mathbf{V})^T \in \mathbb{R}^2$, we get the following system:

$$\epsilon \partial_t \mathbf{U} + A(D) \mathbf{U} = \epsilon F(\zeta, \mathbf{V}),$$

where

$$A(D) = \begin{pmatrix} 0 & 1 \\ \left(1 - \frac{\mu}{3} \Delta\right)^{-1} \Delta & 0 \end{pmatrix} \text{ and } F(\zeta, \mathbf{V}) = - \begin{pmatrix} \nabla \cdot (\zeta \mathbf{V}) \\ \left(1 - \frac{\mu}{3} \Delta\right)^{-1} \nabla \cdot ((\mathbf{V} \cdot \nabla) \mathbf{V}) \end{pmatrix}.$$

Note that

$$\exp(tA(D)) = \begin{pmatrix} \cos\left(t \frac{|D|}{\sqrt{1+\frac{\mu}{3}|D|^2}}\right) & \frac{\sqrt{1+\frac{\mu}{3}|D|^2}}{|D|} \sin\left(t \frac{|D|}{\sqrt{1+\frac{\mu}{3}|D|^2}}\right) \\ -\frac{|D|}{\sqrt{1+\frac{\mu}{3}|D|^2}} \sin\left(t \frac{|D|}{\sqrt{1+\frac{\mu}{3}|D|^2}}\right) & \cos\left(t \frac{|D|}{\sqrt{1+\frac{\mu}{3}|D|^2}}\right) \end{pmatrix}.$$

As in the one-dimensional (1D) case, we expect $(\zeta, \nabla \cdot \mathbf{V})$ to weakly converge to 0. We, however, have no control over the rotational component of \mathbf{V} . Applying the operator $\nabla^\perp \cdot$ to the second equation and denoting by $\omega := \nabla^\perp \cdot \mathbf{V}$ we get the following equation:

$$\partial_t \omega + (\mathbf{V} \cdot \nabla) \omega + (\nabla \cdot \mathbf{V}) \omega = 0.$$

Since \mathbf{V} is a vector field on \mathbb{R}^2 it has a Hodge–Weyl decomposition

$$\mathbf{V} = \nabla \frac{\nabla \cdot \mathbf{V}}{\Delta} + \nabla^\perp \frac{\nabla^\perp \cdot \mathbf{V}}{\Delta}.$$

Therefore we expect ω to converge to $\tilde{\omega}$ as $\epsilon \rightarrow 0$, where

$$\partial_t \tilde{\omega} + \left(\nabla^\perp \frac{\nabla^\perp \cdot \tilde{\omega}}{\Delta} \cdot \nabla \right) \tilde{\omega} = 0.$$

The previous equation is the vorticity formulation of the incompressible Euler equation

$$\partial_t \tilde{\mathbf{V}} + (\tilde{\mathbf{V}} \cdot \nabla) \tilde{\mathbf{V}} + \nabla P = 0, \nabla \cdot \tilde{\mathbf{V}} = 0 \tag{9}$$

with $\tilde{\omega} = \nabla^\perp \cdot \tilde{\mathbf{V}}$. One can now state the main results of this section.

Theorem 2. *Let $M > 0, T > 0, \epsilon \in (0, 1]$ and $\mu \in (0, 1]$. Let $(\zeta, \mathbf{V}) \in C([0, T]; (H^6 \times H^6)(\mathbb{R}^2))$ be a solution of (8) with initial datum (ζ_0, \mathbf{V}_0) and $\tilde{\mathbf{V}} \in C([0, T]; L^2(\mathbb{R}^2))$ be a solution of the incompressible Euler equation (9) with initial datum $\nabla^\perp \frac{\nabla^\perp \cdot \mathbf{V}_0}{\Delta}$ such that*

$$\|(\zeta, \mathbf{V})\|_{L^\infty(0, T; (H^6 \times H^6)(\mathbb{R}^2))} + \|\tilde{\mathbf{V}}\|_{L^\infty(0, T; L^2(\mathbb{R}^2))} \leq M.$$

There exists a constant $C > 0$ independent of M, T, ϵ , and μ such that for any $q, r \geq 2$ with $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$

$$\left\| \left(\begin{array}{c} \zeta \\ \frac{\nabla \cdot \mathbf{V}}{\Delta} \end{array} \right) - \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \exp\left(-\frac{t}{\epsilon} A(D)\right) \left(\begin{array}{c} \zeta_0 \\ \nabla \cdot \mathbf{V}_0 \end{array} \right) \right\|_{L_t^q(0, T; L_x^r(\mathbb{R}^2))} \leq \left(\frac{\epsilon}{\sqrt{\mu}} \ln\left(1 + \frac{\mu}{\epsilon^2} T\right) \right)^{\frac{1}{2} + \frac{1}{q}} M^2 T^{\frac{1}{2}} C + \frac{\epsilon}{\sqrt{\mu}} M^2 T C,$$

$$\left\| \left(\begin{array}{c} \zeta \\ \frac{\nabla \cdot \mathbf{V}}{\Delta} \end{array} \right) \right\|_{L_t^q(0, T; L_x^r(\mathbb{R}^2))} \leq \left(\frac{\epsilon}{\sqrt{\mu}} \ln\left(1 + \frac{\mu}{\epsilon^2} T\right) \right)^{\frac{1}{q}} \left(M + M^2 \left(\frac{\epsilon}{\sqrt{\mu}} \ln\left(1 + \frac{\mu}{\epsilon^2} T\right) T \right)^{\frac{1}{2}} \right) C + \frac{\epsilon}{\sqrt{\mu}} (M + M^2 T) C,$$

$$\left\| \nabla^\perp \frac{\nabla^\perp \cdot \mathbf{V}}{\Delta} - \tilde{\mathbf{V}} \right\|_{L_t^\infty(0, T; L_x^2(\mathbb{R}^2))} \leq \left(\frac{\epsilon}{\sqrt{\mu}} \ln\left(1 + \frac{\mu}{\epsilon^2} T\right) \right)^{\frac{1}{2}} \left(M + M^2 \left(\frac{\epsilon}{\sqrt{\mu}} \ln\left(1 + \frac{\mu}{\epsilon^2} T\right) T \right)^{\frac{1}{2}} \right) M \sqrt{T} e^{CMT} C + \frac{\epsilon}{\sqrt{\mu}} (M + M^2 T) M \sqrt{T} e^{CMT} C.$$

Remark 3. $(\zeta, \frac{\nabla}{\Delta} \nabla \cdot \mathbf{V})$ weakly converges to 0 as $\epsilon \rightarrow 0$, whereas $\frac{\nabla^\perp}{\Delta} \nabla^\perp \cdot \mathbf{V}$ strongly converges to a solution of the incompressible Euler equation. Therefore, \mathbf{V} strongly converges to a solution of the incompressible Euler equation if and only if $(\zeta_0, \nabla \cdot \mathbf{V}_0) = (0, 0)$. The default of strong convergence is exhibited through the first bound of the theorem. Finally, unlike the 1D case, if $\epsilon \sim \mu$ as in Refs. 19 and 20, the theorem provides a convergence result as $\epsilon \rightarrow 0$.

The previous theorem does not provide uniform bounds with respect to $\mu \rightarrow 0$. It is the purpose of the following theorem.

Theorem 3. *Let $M > 0, T > 0, \epsilon \in (0, 1]$, and $\mu \in (0, 1]$. Let $(\zeta, \mathbf{V}) \in C([0, T]; (H^6 \times H^6)(\mathbb{R}^2))$ be a solution of (8) with initial datum (ζ_0, \mathbf{V}_0) , and $\tilde{\mathbf{V}} \in C([0, T]; L^2(\mathbb{R}^2))$ be a solution of the incompressible Euler equation (9) with initial datum $\nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V}_0$ such that*

$$\|(\zeta, \mathbf{V})\|_{L^\infty(0,T;(H^6 \times H^6)(\mathbb{R}^2))} + \|\tilde{\mathbf{V}}\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} \leq M.$$

There exists a constant $C > 0$ independent of M, T, ϵ , and μ such that for any $q, r \geq 2$ with $\frac{1}{q} + \frac{1}{2r} = \frac{1}{4}$

$$\begin{aligned} & \left\| \left(\frac{\nabla \nabla}{\Delta} \cdot \mathbf{V} \right) - \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial_1}{\Delta} \\ 0 & \frac{\partial_2}{\Delta} \end{pmatrix} \exp\left(-\frac{t}{\epsilon} A(D)\right) \begin{pmatrix} \zeta_0 \\ \nabla \cdot \mathbf{V}_0 \end{pmatrix} \right\|_{L_t^q(0,T;L_x^r(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{4} + \frac{1}{q}} M^2 T^{\frac{3}{4}} C, \\ & \left\| \left(\frac{\nabla \nabla}{\Delta} \cdot \mathbf{V} \right) \right\|_{L_t^q(0,T;L_x^r(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{q}} \left(M + M^2 T^{\frac{3}{4}} \epsilon^{\frac{1}{4}} \right) C, \\ & \left\| \nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V} - \tilde{\mathbf{V}} \right\|_{L_t^\infty(0,T;L_x^2(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{4}} \left(M + M^2 T^{\frac{3}{4}} \epsilon^{\frac{1}{4}} \right) M T^{\frac{3}{4}} e^{CMT} C, \\ & \sup_{x_0 \in \mathbb{R}^2} \left\| e^{-(x-x_0)^2} \left(\frac{\nabla \nabla}{\Delta} \cdot \mathbf{V} \right) \right\|_{L_t^2(0,T;L_x^2(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{2}} (M + M^2 T) C. \end{aligned}$$

Remark 4. The last two estimates provide the convergence in $L_t^2(0, T; L_{loc}^2(\mathbb{R}^2))$. We then get the convergence of the local energy to the local energy of the corresponding solution of the incompressible Euler equation. If furthermore $\tilde{\mathbf{V}} \in C([0, T]; H^6(\mathbb{R}^2))$, then the convergence is in $L_t^2(0, T; H_{loc}^s(\mathbb{R}^2))$ for any $s \in [0, 6)$.

3.1 | Local existence

In this section, we provide a local well-posedness result of (8) on an existence time independent of $\epsilon, \mu \in (0, 1]$. Let $k \in \mathbb{N}$ with $k \geq 3$. We introduce the functional space

$$X_\mu^k(\mathbb{R}^2) := \left\{ (\zeta, V) \in (H^k \times H^k)(\mathbb{R}^2), \|(\zeta, V)\|_{X_\mu^k} := \|\zeta\|_{H^k} + \|V\|_{H^k} + \sqrt{\mu} \|\nabla \cdot V\|_{H^k} < \infty \right\}.$$

Proposition 2. *Let $k \in \mathbb{N}, k \geq 3$. Let $h_0 > 0, A > 0, \epsilon \in (0, 1]$, and $\mu \in (0, 1]$. Let $(\zeta_0, V_0) \in X_\mu^k(\mathbb{R}^2)$ with $1 + \epsilon\zeta_0 \geq h_0$ and $\|(\zeta_0, V_0)\|_{X_\mu^k} \leq A$. There exists a time $T > 0$ that only depends on A and h_0 and a unique solution $(\zeta, V) \in C([0, T]; X_\mu^k(\mathbb{R}^2))$ of (8) with initial datum (ζ_0, V_0) . Furthermore, there exists a constant $M > 0$ depending only on A and h_0 such that*

$$\|(\zeta, V)\|_{L^\infty(0, T; X_\mu^k)} \leq M.$$

The proof is similar to the proofs of Ref. [21, Proposition 6.7] or Ref. [25, Proposition 2.7]. Again the key point is that System (8) under the assumption that $(1 + \epsilon\zeta) \geq \frac{h_0}{2}$ has a symmetrizer and corresponding energies that allow to control the $X_\mu^k(\mathbb{R}^2)$ norm.

3.2 | Dispersive estimates

From the properties (4) together with the fact $|g'''(r)| \lesssim r^{-5}$, and thanks to Appendix C one can prove several dispersive estimates.

First, using Lemma C2# together with Lemma B1, for any $m \in \mathbb{N}$ there exists a constant $C > 0$ such that for any Schwartz class function f , any $t \neq 0$, any $\mu \in (0, 1]$ and any $\epsilon \in (0, 1]$ we have

$$\left\| e^{\pm i \frac{t}{\epsilon} \frac{|D|}{\sqrt{1 + \frac{\mu}{3} D^2}}} \frac{\nabla^m}{|D|^m} f \right\|_{L_x^\infty} \leq \frac{C}{\sqrt{\mu}} \frac{\epsilon}{|t|} \|(1 + \mu|D|^2)^2 f\|_{L^1}.$$

Then corresponding Strichartz estimates can be obtained from a T^*T argument: for any $q, r, \tilde{q}, \tilde{r} \geq 2$ such that $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}, \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} = \frac{1}{2}$ with $r, \tilde{r} < \infty$, any $m \in \mathbb{N}$, any $\epsilon \in (0, 1]$ and any $\mu \in (0, 1]$

$$\begin{aligned} \left\| e^{\pm i \frac{t}{\epsilon} \frac{|D|}{\sqrt{1 + \frac{\mu}{3} D^2}}} \frac{\nabla^m}{|D|^m} f \right\|_{L_t^q L_x^r} &\lesssim \left(\frac{\epsilon}{\sqrt{\mu}} \right)^{\frac{1}{2} - \frac{1}{r}} \|(1 + \mu|D|^2)^{2(\frac{1}{2} - \frac{1}{r})} f\|_{L^2} \\ \left\| \int_0^t e^{\pm i \frac{(t-s)}{\epsilon} \frac{|D|}{\sqrt{1 + \frac{\mu}{3} D^2}}} \frac{\nabla^m}{|D|^m} F(s) ds \right\|_{L_t^q L_x^r} &\lesssim \left(\frac{\epsilon}{\sqrt{\mu}} \right)^{1 - \frac{1}{\tilde{r}} - \frac{1}{r}} \|(1 + \mu|D|^2)^{2(1 - \frac{1}{\tilde{r}} - \frac{1}{r})} F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}. \end{aligned} \tag{10}$$

It is well known that the previous estimates do not work at the endpoints $(q, r) = (2, \infty)$ or $(\tilde{q}, \tilde{r}) = (2, \infty)$. We can, however, prove a logarithmic estimate for functions whose Fourier transform is well localized. Such types of estimates were performed for the wave equation (Ref. 26 or Ref. [27, Theorem 8.30]) or the Schrödinger equation.²⁸ We provide in the following a general result.

Proposition 3. *Let χ be a smooth compactly supported function, g a function defined on \mathbb{R}_+^* , $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$. Assume there exists a constant $C_0 > 0$ such that for any Schwartz class function f and*

with $\beta = 1, s = \alpha = -4$.

any $t \neq 0$

$$\left\| e^{\pm itg(|D|)} \frac{\nabla^m}{|D|^m} f \right\|_{L_x^\infty} \leq \frac{C_0}{|t|} \|(1 + |D|^2)^{\frac{\alpha}{2}} f\|_{L^1}.$$

Then there exists a constant $C > 0$ such that for any $\lambda > 0$, any $T > 0$, any $\mu > 0$ and any $\epsilon > 0$ we have

$$\left\| e^{\pm i \frac{t}{\epsilon\sqrt{\mu}} g(\sqrt{\mu}|D|)} \chi(\lambda^{-1}|D|) \frac{\nabla^m}{|D|^m} f \right\|_{L_t^2(0,T;L_x^\infty)} \leq C \left(\frac{\epsilon}{\sqrt{\mu}} \right)^{\frac{1}{2}} \sqrt{\ln \left(1 + \frac{\sqrt{\mu}}{\epsilon} \lambda^2 T \right)} \|(1 + \mu|D|^2)^{\frac{\alpha}{4}} f\|_{L^2}$$

and if we define the operator H as

$$H(F) := \int_0^t e^{\pm i \frac{(t-s)}{\epsilon\sqrt{\mu}} g(\sqrt{\mu}|D|)} \chi(\lambda^{-1}|D|) \frac{\nabla^m}{|D|^m} F(s, \cdot) ds$$

we have

$$\|H(F)\|_{L_t^\infty(0,T;L_x^2)} \leq C \left(\frac{\epsilon}{\sqrt{\mu}} \right)^{\frac{1}{2}} \sqrt{\ln \left(1 + \frac{\sqrt{\mu}}{\epsilon} \lambda^2 T \right)} \|(1 + \mu|D|^2)^{\frac{\alpha}{4}} F\|_{L_t^2(0,T;L_x^1)},$$

$$\|H(F)\|_{L_t^2(0,T;L_x^\infty)} \leq C \left(\frac{\epsilon}{\sqrt{\mu}} \right)^{\frac{1}{2}} \sqrt{\ln \left(1 + \frac{\sqrt{\mu}}{\epsilon} \lambda^2 T \right)} \|(1 + \mu|D|^2)^{\frac{\alpha}{4}} F\|_{L_t^1(0,T;L_x^2)},$$

$$\|H(F)\|_{L_t^2(0,T;L_x^\infty)} \leq C \frac{\epsilon}{\sqrt{\mu}} \ln \left(1 + \frac{\sqrt{\mu}}{\epsilon} \lambda^2 T \right) \|(1 + \mu|D|^2)^{\frac{\alpha}{2}} F\|_{L_t^2(0,T;L_x^1)}.$$

Proof. By assumption and change of variables, we actually have for any $\mu \in (0, 1]$ and any $\epsilon \in (0, 1]$

$$\left\| e^{\pm i \frac{t}{\epsilon\sqrt{\mu}} g(\sqrt{\mu}|D|)} \frac{\nabla^m}{|D|^m} f \right\|_{L_x^\infty} \leq \frac{\epsilon}{\sqrt{\mu}} \frac{C_0}{|t|} \|(1 + \mu|D|^2)^{\frac{\alpha}{2}} f\|_{L^1}. \tag{11}$$

We only prove the case $m = 0$ since the methodology is the same for the other cases. We introduce the operator

$$\begin{aligned} S &: L_t^2(0, T, L_x^1(\mathbb{R}^n)) \rightarrow L^2(\mathbb{R}^2) \\ F &\mapsto \int_0^T e^{\mp i \frac{s}{\epsilon\sqrt{\mu}} g(\sqrt{\mu}|D|)} (1 + \mu|D|^2)^{-\frac{\alpha}{4}} \chi(\lambda^{-1}|D|) F(s) ds. \end{aligned}$$

For any $f \in L^2(\mathbb{R}^2)$

$$\begin{aligned} \|S^* f\|_{L_t^2(0,T;L_x^\infty)} &\leq \sup_{\|G\|_{L_t^2 L_x^1} \leq 1} \left\{ \int_0^T \left\langle e^{\mp i \frac{t}{\epsilon\sqrt{\mu}} g(\sqrt{\mu}|D|)} (1 + \mu|D|^2)^{-\frac{\alpha}{4}} \chi(\lambda^{-1}|D|) f, G(t, \cdot) \right\rangle_{L_x^2 \times L_x^2} dt \right\} \\ &\leq \|f\|_{L^2} \sup_{\|G\|_{L_t^2(0,T;L_x^1)} \leq 1} J_G \end{aligned}$$

where

$$J_G := \left\{ \left\| \int_0^T e^{\mp i \frac{t}{\epsilon \sqrt{\mu}} g(\sqrt{\mu}|D|)} (1 + \mu|D|^2)^{-\frac{\alpha}{4}} \chi(\lambda^{-1}|D|) G(t, \cdot) dt \right\|_{L_x^\infty} \right\}.$$

Then

$$J_G^2 \leq \int_{[0,T]^2} \underbrace{\left\| e^{\mp i \frac{(t-s)}{\epsilon \sqrt{\mu}} g(\sqrt{\mu}|D|)} (1 + \mu|D|^2)^{-\frac{\alpha}{2}} \chi^2(\lambda^{-1}|D|) G(t, \cdot) \right\|_{L_x^\infty}}_{:=K_G} \|G(s, \cdot)\|_{L_x^1} dt ds$$

and using (11)

$$K_G \lesssim \frac{\epsilon}{\sqrt{\mu}} \frac{1}{|t-s|} \|G(t, \cdot)\|_{L_x^1}$$

whereas from Bernstein’s Lemma B2

$$\begin{aligned} K_G &\lesssim \lambda \left\| e^{\mp i \frac{(t-s)}{\epsilon \sqrt{\mu}} g(\sqrt{\mu}|D|)} (1 + \mu|D|^2)^{-\frac{\alpha}{2}} \chi(\lambda^{-1}|D|) G(t, \cdot) \right\|_{L_x^2} \\ &\lesssim \lambda \|\chi^2(\lambda^{-1}|D|) G(t, \cdot)\|_{L_x^2} \lesssim \lambda^2 \|G(t, \cdot)\|_{L_x^1} \end{aligned}$$

so that

$$J_G^2 \leq \int_{[0,T]^2} \frac{\lambda^2}{1 + \lambda^2 \frac{\sqrt{\mu}}{\epsilon} |t-s|} \|G(t, \cdot)\|_{L_x^1} \|G(s, \cdot)\|_{L_x^1} dt ds.$$

The first bound follows from Schur’s test. Then for any $(a, b), (\bar{a}, \bar{b}) \in \{(2, \infty), (\infty, 2)\}$, we define the operator $R_{\bar{a}, a}$

$$\begin{aligned} R_{\bar{a}, a} &: L_t^{\bar{b}'}(\mathbb{R}, L_x^{\bar{a}'}(\mathbb{R}^2)) \rightarrow L_t^b(\mathbb{R}, L_x^a(\mathbb{R}^2)) \\ F &\mapsto \int_0^t e^{\pm i \frac{(t-s)}{\epsilon \sqrt{\mu}} g(\sqrt{\mu}|D|)} (1 + \mu|D|^2)^{-\frac{\alpha}{4}} \chi(\lambda^{-1}|D|) F(s, \cdot) ds. \end{aligned}$$

We note that

$$\|R_{\infty, 2}(F)\|_{L_t^\infty(0, T; L_x^2)} = \sup_{t \in [0, T]} \|S(\mathbb{1}_{(0, t)} F)\|_{L_x^2}$$

so that the second bound is followed by duality and the first bound. Furthermore, we notice that $R_{2, \infty} + R_{\infty, 2}^* = S^* \tilde{S}$ where

$$\begin{aligned} \tilde{S} &: L_t^1(0, T, L_x^2(\mathbb{R}^n)) \rightarrow L^2(\mathbb{R}^2) \\ F &\mapsto \int_0^T e^{\mp i \frac{s}{\epsilon \sqrt{\mu}} g(\sqrt{\mu}|D|)} F(s) ds. \end{aligned}$$

The third estimate follows from the first and second estimates together with the fact that \tilde{S} is bounded since \tilde{S}^* is bounded. Finally, denoting

$$L_F := \left\| \int_0^t e^{\pm i \frac{(t-s)}{\epsilon \sqrt{\mu}} g(\sqrt{\mu}|D|)} (1 + \mu|D|^2)^{-\frac{\alpha}{2}} \chi(\lambda^{-1}|D|) F(s, \cdot) ds \right\|_{L_t^2(0,T;L_x^\infty)}$$

and proceeding similarly as for the bound on K_G we get

$$\begin{aligned} L_F &\lesssim \left\| \int_{s=0}^t \frac{\lambda^2}{1 + \lambda^2 \frac{\sqrt{\mu}}{\epsilon} |t-s|} \|F(s, \cdot)\|_{L_x^1} ds \right\|_{L_t^2(0,T)} \\ &\lesssim \sqrt{\sup_{t \in [0,T]} \int_{s=0}^t \frac{\lambda^2}{1 + \lambda^2 \frac{\sqrt{\mu}}{\epsilon} |t-s|} ds} \sqrt{\int_{t=0}^T \int_{s=0}^t \frac{\lambda^2}{1 + \lambda^2 \frac{\sqrt{\mu}}{\epsilon} |t-s|} \|F(s, \cdot)\|_{L_x^1}^2 ds dt} \end{aligned}$$

and the fourth bound follows. □

Second, we get from Lemma C3(i)^{||} together with Lemma B1 that if χ is a smooth compactly supported that is equal to 1 near 0 and $m \in \mathbb{N}$

$$\left\| e^{\pm i \frac{t}{\epsilon} \frac{|D|}{\sqrt{1+\frac{\mu}{3}D^2}}} \chi(\sqrt{\mu}|D|) \frac{\nabla^m}{|D|^m} f \right\|_{L_x^\infty} \lesssim \frac{\epsilon^{\frac{1}{2}}}{|t|^{\frac{1}{2}}} \|(1 + |D|^2) f\|_{L^1}$$

and from Lemma C2(ii)^{**} together with Lemma B1 that

$$\left\| e^{\pm i \frac{t}{\epsilon} \frac{|D|}{\sqrt{1+\frac{\mu}{3}D^2}}} (1 - \chi(\sqrt{\mu}|D|)) \frac{\nabla^m}{|D|^m} f \right\|_{L_x^\infty} \lesssim \mu^{\frac{3}{4}} \frac{\epsilon^{\frac{1}{2}}}{|t|^{\frac{1}{2}}} \|(1 + |D|^2)^{\frac{3}{2}} f\|_{L^1}.$$

One can then obtain corresponding Strichartz estimates from a T^*T argument: for any $(\tilde{q}, \tilde{r}) \in \{(4, \infty), (\infty, 2)\}$, for any $\mu \in (0, 1]$, any $\epsilon \in (0, 1]$ and any $m \in \mathbb{N}$

$$\begin{aligned} \left\| e^{\pm i \frac{t}{\epsilon} \frac{|D|}{\sqrt{1+\frac{\mu}{3}D^2}}} \frac{\nabla^m}{|D|^m} f \right\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} &\lesssim \epsilon^{\frac{1}{2}(\frac{1}{2}-\frac{1}{\tilde{r}})} \|(1 + |D|^2)^{\frac{3}{2}(\frac{1}{2}-\frac{1}{\tilde{r}})} f\|_{L^2}, \\ \left\| \int_0^t e^{\pm i \frac{t-s}{\epsilon} \frac{|D|}{\sqrt{1+\frac{\mu}{3}D^2}}} \frac{\nabla^m}{|D|^m} F(s) ds \right\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} &\lesssim \epsilon^{\frac{1}{2}(1-\frac{1}{\tilde{r}})} \|(1 + |D|^2)^{\frac{3}{2}(1-\frac{1}{\tilde{r}})} F\|_{L_t^{\frac{4}{3}} L_x^1}. \end{aligned} \tag{12}$$

^{||} with $l = 2, \eta = \frac{1}{2}$.

^{**} with $s = -3, \alpha = -4$.

3.3 | Proofs of Theorems 2 and 3

We begin with the proof of Theorem 2. Using Duhamel’s principle

$$\left(\begin{array}{c} \zeta \\ \frac{\nabla}{\Delta} \cdot \mathbf{V} \end{array} \right) - \underbrace{\left(\begin{array}{cc} 1 & 0 \\ 0 & \frac{\partial_1}{\Delta} \\ 0 & \frac{\partial_2}{\Delta} \end{array} \right) \exp(-tA(D)) \left(\begin{array}{c} \zeta_0 \\ \nabla \cdot \mathbf{V}_0 \end{array} \right)}_{:=I} = \int_0^t \underbrace{\left(\begin{array}{cc} 1 & 0 \\ 0 & \frac{\partial_1}{\Delta} \\ 0 & \frac{\partial_2}{\Delta} \end{array} \right) \exp\left(\frac{\tau-t}{\epsilon}A(D)\right)}_{:=I} F(\zeta(\tau), \mathbf{V}(\tau)) d\tau.$$

Let $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ and χ be a smooth compactly supported function that is equal to 1 near 0. By Proposition 3 and interpolation, for any $j \in \{1, 2\}$

$$\left\| \chi \left(\sqrt{\frac{\epsilon}{\sqrt{\mu}}} |D| \right) I_j \right\|_{L_t^q(0,T;L_x^r(\mathbb{R}^2))} \lesssim \left(\frac{\epsilon}{\sqrt{\mu}} \ln \left(1 + \frac{\mu}{\epsilon^2} T \right) \right)^{\frac{1}{q} + \frac{1}{2}} B_j$$

where

$$B_1 := \left\| (1 + \mu|D|^2)^{2(1-\frac{1}{r})} \nabla \cdot (\zeta \mathbf{V}) \right\|_{L_t^2(0,T;L_x^1)} + \left\| \frac{(1 + \mu|D|^2)^{2(1-\frac{1}{r})}}{\sqrt{1 + \frac{\mu}{3}|D|^2}} ((\mathbf{V} \cdot \nabla) \mathbf{V}) \right\|_{L_t^2(0,T;L_x^1)}$$

$$B_2 := \left\| \frac{(1 + \mu|D|^2)^{2(1-\frac{1}{r})}}{\sqrt{1 + \frac{\mu}{3}|D|^2}} \nabla \cdot (\zeta \mathbf{V}) \right\|_{L_t^2(0,T;L_x^1)} + \left\| \frac{(1 + \mu|D|^2)^{2(1-\frac{1}{r})}}{1 + \frac{\mu}{3}|D|^2} ((\mathbf{V} \cdot \nabla) \mathbf{V}) \right\|_{L_t^2(0,T;L_x^1)}.$$

Using Lemmas B1 and B5, we get

$$B_1 + B_2 \lesssim \| \zeta \|_{H_x^5} \| \mathbf{V} \|_{H_x^5} \| \cdot \|_{L_t^2(0,T)} + \| \mathbf{V} \|_{H_x^5}^2 \| \cdot \|_{L_t^2(0,T)} \lesssim T^{\frac{1}{2}} M^2.$$

Furthermore, using Lemma B3

$$\begin{aligned} \left\| (1 - \chi \left(\sqrt{\frac{\epsilon}{\sqrt{\mu}}} |D| \right)) I_j \right\|_{L_t^\infty(0,T;L_x^2(\mathbb{R}^2))} &\lesssim \frac{\epsilon}{\sqrt{\mu}} \| |D|^2 I_j \|_{L_t^\infty(0,T;L_x^2(\mathbb{R}^2))} \\ &\lesssim \frac{\epsilon}{\sqrt{\mu}} \left(\| \zeta \|_{H_x^3} \| \mathbf{V} \|_{H_x^3} \| \cdot \|_{L_t^1(0,T)} + \| \mathbf{V} \|_{H_x^3}^2 \| \cdot \|_{L_t^1(0,T)} \right) \\ &\lesssim T \frac{\epsilon}{\sqrt{\mu}} M^2 \end{aligned}$$

whereas from Sobolev inequalities, Lemma B3 and (10) with $r = \tilde{r} = \frac{1}{4}$

$$\begin{aligned} \left\| \left(1 - \chi \left(\sqrt{\frac{\epsilon}{\sqrt{\mu}}} |D| \right) \right) I_j \right\|_{L_t^2(0,T;L_x^\infty(\mathbb{R}^2))} &\lesssim \left\| \left(1 - \chi \left(\sqrt{\frac{\epsilon}{\sqrt{\mu}}} |D| \right) \right) I_j \right\|_{L_t^2(0,T;W_x^{1,4}(\mathbb{R}^2))} \\ &\lesssim T^{\frac{1}{4}} \left\| \left(1 - \chi \left(\sqrt{\frac{\epsilon}{\sqrt{\mu}}} |D| \right) \right) I_j \right\|_{L_t^4(0,T;W_x^{1,4}(\mathbb{R}^2))} \\ &\lesssim T^{\frac{1}{4}} \left(\frac{\epsilon}{\sqrt{\mu}} \right)^{\frac{1}{2}} \left\| I_j \right\|_{L_t^4(0,T;W_x^{2,4}(\mathbb{R}^2))} \\ &\lesssim T^{\frac{1}{4}} \frac{\epsilon}{\sqrt{\mu}} \left(\left\| \zeta \right\|_{H_x^5} \left\| \mathbf{V} \right\|_{H_x^5} \right)^{\frac{4}{3}} + \left\| \mathbf{V} \right\|_{H_x^5}^2 \left\| \cdot \right\|_{L_t^{\frac{4}{3}}(0,T)} \\ &\lesssim T \frac{\epsilon}{\sqrt{\mu}} M^2. \end{aligned}$$

The first bound follows by Hölder’s inequality. One can similarly obtain the second bound. Note that by differentiating one time we can also get a bound on $\left\| \nabla_{\Delta}^{\perp} \cdot \mathbf{V} \right\|_{L_t^2(0,T;W_x^{1,\infty}(\mathbb{R}^2))}$ which will be useful in the following.

On the other hand, since $\tilde{\mathbf{V}} = \nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot \tilde{\mathbf{V}}$,

$$\partial_t \left(\nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot \mathbf{v} - \tilde{\mathbf{V}} \right) + \nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot ((\tilde{\mathbf{V}} \cdot \nabla)(\mathbf{v} - \tilde{\mathbf{V}})) + \nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot ((\mathbf{v} - \tilde{\mathbf{V}}) \cdot \nabla) \mathbf{v} = 0$$

so that integrating by parts

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\left\| \nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot \mathbf{v} - \tilde{\mathbf{V}} \right\|_{L_x^2}^2 \right) &= - \underbrace{\int_{\mathbb{R}^2} ((\tilde{\mathbf{V}} \cdot \nabla)(\mathbf{v} - \tilde{\mathbf{V}})) \cdot \left(\nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot \mathbf{v} - \tilde{\mathbf{V}} \right)}_{:=J_1} \\ &\quad - \underbrace{\int_{\mathbb{R}^2} (((\mathbf{v} - \tilde{\mathbf{V}}) \cdot \nabla) \mathbf{v}) \cdot \left(\nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot \mathbf{v} - \tilde{\mathbf{V}} \right)}_{:=J_2}. \end{aligned}$$

Then

$$J_1 = \int_{\mathbb{R}^2} \left((\tilde{\mathbf{V}} \cdot \nabla) \left(\nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot \mathbf{v} - \tilde{\mathbf{V}} \right) \right) \cdot \left(\nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot \mathbf{v} - \tilde{\mathbf{V}} \right) + \int_{\mathbb{R}^2} \left((\tilde{\mathbf{V}} \cdot \nabla) \nabla \frac{\nabla}{\Delta} \cdot \mathbf{v} \right) \cdot \left(\nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot \mathbf{v} - \tilde{\mathbf{V}} \right)$$

so that integrating by parts in the first integral and using that $\nabla \cdot \tilde{\mathbf{V}} = 0$

$$|J_1| \leq \left\| \nabla \nabla \frac{\nabla}{\Delta} \cdot \mathbf{v} \right\|_{L_x^\infty} \left\| \tilde{\mathbf{V}} \right\|_{L_x^2} \left\| \nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot \mathbf{v} - \tilde{\mathbf{V}} \right\|_{L_x^2},$$

whereas

$$|J_2| \leq \|\nabla \mathbf{V}\|_{L_x^\infty} \|\nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V} - \tilde{\mathbf{V}}\|_{L_x^2}^2 + \|\nabla \frac{\nabla}{\Delta} \cdot \mathbf{V}\|_{L_x^\infty} \|\nabla \mathbf{V}\|_{L_x^2} \|\nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V} - \tilde{\mathbf{V}}\|_{L_x^2}$$

which yields

$$\frac{d}{dt} \left(\|\nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V} - \tilde{\mathbf{V}}\|_{L_x^2}^2 \right) \leq CM \|\nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V} - \tilde{\mathbf{V}}\|_{L_x^2}^2 + M \|\nabla \frac{\nabla}{\Delta} \cdot \mathbf{V}\|_{W_x^{1,\infty}} \|\nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V} - \tilde{\mathbf{V}}\|_{L_x^2}.$$

Bounds on $\nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V} - \tilde{\mathbf{V}}$ follow from Grönwall’s inequality and previous bounds obtained on $\|\nabla \frac{\nabla}{\Delta} \cdot \mathbf{V}\|_{L_t^2(0,T;W_x^{1,\infty})}$.

Concerning Theorem 3, one can use the previous strategy together with (12).^{††} The last bound is a consequence of Morawetz-type estimates established in Proposition 4.

4 | OTHER ABCD BOUSSINESQ SYSTEMS

In the previous two sections, we chose to present the rigid lid limit on one specific Boussinesq system (the case $a = b = c = 0$ and $d = \frac{1}{3}$). There are other abcd-Boussinesq systems

$$\begin{cases} \epsilon(1 - \mu b \Delta) \partial_t \zeta + \nabla \cdot ([1 + \epsilon \zeta] \mathbf{V}) + \mu a \Delta \nabla \cdot \mathbf{V} = 0, \\ \epsilon(1 - \mu d \nabla \nabla \cdot) \partial_t \mathbf{V} + \nabla \zeta + \epsilon(\mathbf{V} \cdot \nabla) \mathbf{V} + \mu c \Delta \nabla \zeta = 0. \end{cases} \tag{13}$$

In the following, we assume that

$$b \geq 0, d \geq 0, a \leq 0, c \leq 0 \tag{14}$$

in order to get the well-posedness of the system (see, for instance, Ref. 19).

We introduce

$$g(y) = y \sqrt{\frac{(1 - ay^2)(1 - cy^2)}{(1 + by^2)(1 + dy^2)}} \text{ and } R(y) = \sqrt{\frac{(1 - ay^2)(1 + dy^2)}{(1 + by^2)(1 - cy^2)}}.$$

As before, in the 1D case if we denote by $\mathbf{U} = (\zeta, V)^T$ we have the following system:

$$\epsilon \partial_t \mathbf{U} + A(\partial_x) \mathbf{U} = \epsilon F(\zeta, V),$$

where

$$A(\partial_x) = \begin{pmatrix} 0 & (1 - \mu b \partial_x^2)^{-1} (1 + \mu a \partial_x^2) \partial_x \\ (1 - \mu d \partial_x^2)^{-1} (1 + \mu c \partial_x^2) \partial_x & 0 \end{pmatrix}$$

$$F(\zeta, V) = - \begin{pmatrix} (1 - \mu b \partial_x^2)^{-1} \partial_x (\zeta V) \\ (1 - \mu d \partial_x^2)^{-1} \partial_x \left(\frac{1}{2} V^2 \right) \end{pmatrix}.$$

^{††} It is not necessary to split the low and the high frequencies in that case since we do not use Proposition 3.

Note that

$$\exp(tA(\partial_x)) = \begin{pmatrix} \cos\left(\frac{t}{\sqrt{\mu}}g(\sqrt{\mu}D)\right) & R(\sqrt{\mu}D)\sin\left(\frac{t}{\sqrt{\mu}}g(\sqrt{\mu}D)\right) \\ \frac{1}{R(\sqrt{\mu}D)}\sin\left(\frac{t}{\sqrt{\mu}}g(\sqrt{\mu}D)\right) & \cos\left(\frac{t}{\sqrt{\mu}}g(\sqrt{\mu}D)\right) \end{pmatrix}.$$

In the two-dimensional (2D) case if we denote by $\mathbf{U} = (\zeta, \nabla \cdot \mathbf{V})^T$, we get the following system:

$$\epsilon \partial_t \mathbf{U} + A(D)\mathbf{U} = \epsilon F(\zeta, \mathbf{V}),$$

where

$$A(D) = \begin{pmatrix} 0 & (1 - \mu b \Delta)^{-1}(1 + \mu a \Delta) \\ (1 - \mu d \Delta)^{-1}(1 + \mu c \Delta) \Delta & 0 \end{pmatrix}$$

$$F(\zeta, \mathbf{V}) = - \begin{pmatrix} (1 - \mu b \Delta)^{-1} \nabla \cdot (\zeta \mathbf{V}) \\ (1 - \mu d \Delta)^{-1} \nabla \cdot ((\mathbf{V} \cdot \nabla) \mathbf{V}) \end{pmatrix}.$$

Note that

$$\exp(tA(D)) = \begin{pmatrix} \cos\left(\frac{t}{\sqrt{\mu}}g(\sqrt{\mu}|D|)\right) & \frac{R(\sqrt{\mu}|D|)}{|D|}\sin\left(\frac{t}{\sqrt{\mu}}g(\sqrt{\mu}|D|)\right) \\ \frac{|D|}{R(|D|)}\sin\left(\frac{t}{\sqrt{\mu}}g(\sqrt{\mu}|D|)\right) & \cos\left(\frac{t}{\sqrt{\mu}}g(\sqrt{\mu}|D|)\right) \end{pmatrix}.$$

The strategy presented in the previous two sections together with ad hoc dispersive estimates provides similar results for System (13), with a rate of convergence depending on how dispersive System (13) is. The existence of solutions of (13) on an existence time independent of ϵ uniformly with respect to $\mu \in (0, 1]$ can easily be adapted from Refs. 29 and 30.

The phase g satisfies the following properties that are carefully studied in Refs. [31, Section 3.5]. First, if $a + b + c + d \neq 0$

$$g'(r) - 1 \underset{r \sim 0}{\sim} -\frac{3(a + b + c + d)}{2}r^2, g''(r) \underset{r \sim 0}{\sim} -3(a + b + c + d)r$$

whereas if $a + b + c + d = 0$

$$g'(r) - 1 \underset{r \sim 0}{\sim} -\frac{5(a + b)(b + c)}{2}r^4, g''(r) \underset{r \sim 0}{\sim} -10(a + b)(b + c)r^3.$$

Second, there exists $\alpha \in [-6, 1] \cap \mathbb{Z}, \ell, \Lambda_1, \Lambda_2 \in \mathbb{R}$ such that

$$g'(r) - \ell \underset{\infty}{\sim} \Gamma_1 r^{\alpha+1}, g''(r) \underset{\infty}{\sim} (\alpha + 1)\Gamma_1 r^\alpha, g'''(r) \leq \Gamma_2 r^{\alpha-1}.$$

The exact value of α and ℓ (that depends on a, b, c, d) can be found in Refs. [31, Table 1]. Finally, one can prove that $|g'| + |g''| + |g'''| > 0$ on \mathbb{R}^+ (see Ref. [31, Lemma 3.4]).

We can now state our results. We begin with the case $n = 1$ in the case $a + b + c + d \neq 0$.

Theorem 4. *Let $n = 1$. Let a, b, c, d satisfying (14) and $a + b + c + d \neq 0$. Let $M > 0, T > 0, \epsilon \in (0, 1]$ and $\mu \in (0, 1]$. Let $(\zeta, V) \in C([0, T]; (H^3 \times H^3)(\mathbb{R}))$ a solution of (13) with initial datum*

(ζ_0, V_0) such that

$$\|(\zeta, V)\|_{L^\infty(0,T;H^3(\mathbb{R})\times H^3(\mathbb{R}))} \leq M.$$

There exists $p \in \mathbb{N}$ with $p \geq 3$ and a constant $C > 0$ depending only on p and a, b, c, d such that for any $q, r \geq 2$ with $\frac{1}{q} + \frac{1}{pr} = \frac{1}{2p}$

$$\begin{aligned} \left\| \begin{pmatrix} \zeta \\ V \end{pmatrix} - e^{-\frac{t}{\epsilon}A(\partial_x)} \begin{pmatrix} \zeta_0 \\ V_0 \end{pmatrix} \right\|_{L_t^q(0,T;L_x^r(\mathbb{R}))} &\leq \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2p} + \frac{1}{q}} M^2 T^{\frac{2p-1}{2p}} C, \\ \left\| \begin{pmatrix} \zeta \\ V \end{pmatrix} \right\|_{L_t^q(0,T;L_x^r(\mathbb{R}))} &\leq \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{q}} \left(M + M^2 T^{\frac{2p-1}{2p}} \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2p}} \right) C. \end{aligned}$$

Furthermore if $|g''| > 0$ on \mathbb{R}_+^* , for any $\tilde{q}, \tilde{r} \geq 2$ with $\frac{1}{\tilde{q}} + \frac{1}{2\tilde{r}} = \frac{1}{4}$

$$\left\| \begin{pmatrix} \zeta \\ V \end{pmatrix} - e^{-\frac{t}{\epsilon}A(\partial_x)} \begin{pmatrix} \zeta_0 \\ V_0 \end{pmatrix} \right\|_{L_t^{\tilde{q}}(0,T;L_x^{\tilde{r}}(\mathbb{R}))} \leq \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{4} + \frac{1}{\tilde{q}}} M^2 T^{\frac{3}{4}} C.$$

Finally, denoting $p_0 = 1$ if $|g'| > 0$ on \mathbb{R}^+ , $p_0 = 2$ if $|g'| + |g''| > 0$ on \mathbb{R}^+ and $p_0 = 3$ otherwise, we have

$$\sup_{x_0 \in \mathbb{R}} \left\| e^{-(x-x_0)^2} \begin{pmatrix} \zeta \\ V \end{pmatrix} \right\|_{L_t^2(0,T;L_x^2(\mathbb{R}))} \leq \epsilon^{\frac{1}{2p_0}} (M + M^2 T) C.$$

If we denote by m the maximum among the multiplicities of positive zeros of g'' then one can take $p = \max(m + 2, 3)$. The proof of the previous theorem follows from dispersive estimates based on Lemma C1^{‡‡} and the properties of g . We only provide a proof of the last point. Let χ_1 be a smooth bounded function supported on $\{|g'| > 0\}$ and χ_2 a smooth compactly supported function supported on $\{|g''| + |g'''| > 0\}$ with $0 \notin \text{supp}(\chi_2)$. On the one hand, we get from Proposition 4 that

$$\sup_{x_0 \in \mathbb{R}} \left\| e^{-(x-x_0)^2} \chi_1(\sqrt{\mu}|D|) \begin{pmatrix} \zeta \\ V \end{pmatrix} \right\|_{L_t^2(0,T;L_x^2(\mathbb{R}))} \leq \epsilon^{\frac{1}{2}} (M + M^2 T) C.$$

On the other hand, using Lemma C1(ii)^{§§} together with Bernstein's Lemma B2, we have

$$\left\| e^{\pm \frac{it}{\sqrt{\mu}}g(\sqrt{\mu}|D|)} \chi_2(\sqrt{\mu}|D|) f \right\|_{L^\infty} \lesssim \frac{1}{|t|^{\frac{1}{p_0}}} \| |D|^{\frac{p_0-1}{p_0}} \chi_2(\sqrt{\mu}|D|) f \|_{L^1}$$

^{‡‡} with $\beta = 1, s = 0, l = p$ and $\beta = 1, s = \frac{1}{2}, l = 2$ for the third estimate.

^{§§} with $l = p_0, \alpha = 0$.

so that from corresponding Strichartz estimates, we obtain

$$\left\| \chi_2(\sqrt{\mu}|D|) \begin{pmatrix} \zeta \\ V \end{pmatrix} \right\|_{L_t^{2p_0}(0,T;L_x^\infty(\mathbb{R}))} \leq \epsilon^{\frac{1}{2p_0}} \left(M + M^2 T^{\frac{2p_0-1}{2p_0}} \epsilon^{\frac{1}{2p_0}} \right) C.$$

The third point of the theorem follows from the fact that $|g'| + |g''| + |g'''| > 0$ on \mathbb{R}^+ and that there exists $0 < y_0 \leq y_1$ such that $g' > 0$ on $[0, y_0] \cup [y_1, \infty)$.

Remark 5. As noted in Remark 2 concerning Theorem 1, in the case $\epsilon \sim \mu$ as in Refs. 19 and 20 or when $\mu = \mathcal{O}(\epsilon)$, the first estimate of Theorem 4 does not provide a convergence result as $\epsilon \rightarrow 0$ so that nonlinear terms must be taken into account and that asymptotic models like a system of decoupling KdV equations

$$\epsilon \partial_t g_\pm \pm \partial_x g_\pm \pm \mu \frac{a+b+c+d}{6} \partial_x^3 g_\pm \pm \frac{3}{2} \epsilon g_\pm \partial_x g_\pm = 0$$

becomes relevant. A proof of such a result can be adapted from, for instance, Ref. [21, Section 7.3.2] together with the symmetrizers and energy estimates from Refs. 29 and 30.

We now consider the case $n = 1$ in the case $a + b + c + d = 0$. We introduce the condition

$$((a+b)(a+d)(c+b)(c+d))^2 + (a+b+c+d)^2 > 0, \tag{15}$$

which avoids the situation where $g(r) \equiv r$ provides a nondispersive system when $n = 1$.

Theorem 5. *Let $n = 1$. Let a, b, c, d satisfying (14) with $a + b + c + d = 0$. Let $M > 0, T > 0, \epsilon \in (0, 1]$ and $\mu \in (0, 1]$. Let $(\zeta, V) \in C([0, T]; (H^3 \times H^3)(\mathbb{R}))$ a solution of (13) with initial datum (ζ_0, V_0) such that*

$$\|(\zeta, V)\|_{L^\infty(0,T;H^3(\mathbb{R}) \times H^3(\mathbb{R}))} \leq M.$$

If a, b, c, d satisfy (15), there exists $p \in \mathbb{N}$ with $p \geq 5$ and a constant $C > 0$ depending only on p and a, b, c, d such that for any $q, r \geq 2$ with $\frac{1}{q} + \frac{1}{pr} = \frac{1}{2p}$

$$\left\| \begin{pmatrix} \zeta \\ V \end{pmatrix} - e^{-\frac{t}{\epsilon} A(\partial_x)} \begin{pmatrix} \zeta_0 \\ V_0 \end{pmatrix} \right\|_{L_t^q(0,T;L_x^r(\mathbb{R}))} \leq \left(\frac{\epsilon}{\mu^2} \right)^{\frac{1}{2p} + \frac{1}{q}} M^2 T^{\frac{2p-1}{2p}} C,$$

$$\left\| \begin{pmatrix} \zeta \\ V \end{pmatrix} \right\|_{L_t^q(0,T;L_x^r(\mathbb{R}))} \leq \left(\frac{\epsilon}{\mu^2} \right)^{\frac{1}{q}} \left(M + M^2 T^{\frac{2p-1}{2p}} \left(\frac{\epsilon}{\mu^2} \right)^{\frac{1}{2p}} \right) C.$$

Furthermore, if some $l \in \{2, 3, 4\}$, we have $\sum_{k=2}^l |g^{(k)}| > 0$ on \mathbb{R}_+^ and if we denote by $\sigma = \min(\frac{2}{5}, \frac{1}{l})$, for any $\tilde{q}, \tilde{r} \geq 2$ with $\frac{1}{\tilde{q}} + \frac{\sigma}{\tilde{r}} = \frac{\sigma}{2}$*

$$\left\| \begin{pmatrix} \zeta \\ V \end{pmatrix} - e^{-\frac{t}{\epsilon} A(\partial_x)} \begin{pmatrix} \zeta_0 \\ V_0 \end{pmatrix} \right\|_{L_t^{\tilde{q}}(0,T;L_x^{\tilde{r}}(\mathbb{R}))} \leq \left(\frac{\epsilon}{\mu^2} \right)^{\frac{\sigma}{2} + \frac{1}{\tilde{q}}} M^2 T^{\frac{2-\sigma}{2}} C.$$

Finally denoting $p_0 = 1$ if $|g'| > 0$ on \mathbb{R}^+ , $p_0 = 2$ if $|g'| + |g''| > 0$ on \mathbb{R}^+ and $p_0 = 3$ otherwise, we have

$$\sup_{x_0 \in \mathbb{R}} \left\| e^{-(x-x_0)^2} \begin{pmatrix} \zeta \\ V \end{pmatrix} \right\|_{L_t^2(0,T;L_x^2(\mathbb{R}))} \leq \epsilon^{\frac{1}{2p_0}} (M + M^2 T) C.$$

If we denote by m the maximum among the multiplicities of positive zeros of g'' then one can take $p = \max(m + 2, 5)$. Again, one can obtain dispersive estimates thanks to Lemma C1^{¶¶} and the previous properties on the phase g . Note that the ratio $\frac{\epsilon}{\mu^2}$ comes from low-frequency estimates: if χ is a smooth compactly supported function whose support is small enough and that is equal to 1 near 0 and if $a + b + c + d = 0$

$$\begin{aligned} \left\| e^{\pm \frac{it}{\epsilon\sqrt{\mu}} g(\sqrt{\mu}|D|)} \chi(\sqrt{\mu}|D|) \partial_x f \right\|_{L^\infty} &\lesssim \frac{1}{|t|^{\frac{1}{5}}} \left(\frac{\epsilon}{\mu^2} \right)^{\frac{2}{5}} \|f\|_{L^1}, \\ \left\| e^{\pm \frac{it}{\epsilon\sqrt{\mu}} g(\sqrt{\mu}|D|)} \chi(\sqrt{\mu}|D|) f \right\|_{L^\infty} &\lesssim \frac{1}{|t|^{\frac{1}{5}}} \left(\frac{\epsilon}{\mu^2} \right)^{\frac{1}{5}} \|f\|_{L^1}. \end{aligned}$$

Remark 6. When $\epsilon \sim \mu$ as in Refs. 19 and 20 or when $\mu = \mathcal{O}(\epsilon)$, the first estimate of Theorem 4 does not provide a convergence result as $\epsilon \rightarrow 0$. Again, nonlinear terms must be taken into account. Note, however, that here one must consider a system of decoupling Burgers equations

$$\epsilon \partial_t g_{\pm} \pm \partial_x g_{\pm} \pm \frac{3}{2} \epsilon g_{\pm} \partial_x g_{\pm} = 0.$$

A proof of such a result can be adapted from, for instance, Ref. [21, Section 7.3.2] together with the symmetrizers and energy estimates from Refs. 29 and 30.

We now consider the case $n = 2$ with $a + b + c + d \neq 0$.

Theorem 6. *Let $n = 2$. Let a, b, c, d satisfying (14) with $a + b + c + d \neq 0$. Let $M > 0, T > 0, \epsilon \in (0, 1]$, and $\mu \in (0, 1]$. Let $(\zeta, \mathbf{V}) \in C([0, T]; (H^6 \times H^6)(\mathbb{R}^2))$ be a solution of (13) with initial datum (ζ_0, \mathbf{V}_0) , and let $\tilde{\mathbf{V}} \in C([0, T]; L^2(\mathbb{R}^2))$ be a solution of the incompressible Euler equation (9) with initial datum $\nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V}_0$ such that*

$$\|(\zeta, \mathbf{V})\|_{L^\infty(0,T;(H^6 \times H^6)(\mathbb{R}^2))} + \|\tilde{\mathbf{V}}\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} \leq M.$$

-If g' and g'' do not vanish on \mathbb{R}_+^ , (ζ, \mathbf{V}) satisfy the same estimates as in Theorem 2.*

^{¶¶} with $\beta = 3, s = 0, l = p$ and $\beta = 3, s = 5\sigma - 1$ for the third estimate.

If g' does not vanish on \mathbb{R}^+ but g'' vanishes on \mathbb{R}_+^* , there exists $\sigma \in (\frac{1}{2}, 1)$ and a constant $C > 0$ depending only on a, b, c, d such that for any $q, r \geq 2$ with $\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}$

$$\begin{aligned} & \left\| \left(\frac{\nabla \nabla}{\Delta} \cdot \mathbf{v} \right) - \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial_1}{\Delta} \\ 0 & \frac{\partial_2}{\Delta} \end{pmatrix} \exp\left(-\frac{t}{\epsilon} A(D)\right) \left(\nabla \cdot \mathbf{v}_0 \right) \right\|_{L_t^q(0, T; L_x^r(\mathbb{R}^2))} \leq \left(\frac{\epsilon}{\sqrt{\mu}} \right)^{\frac{\sigma}{2} + \frac{1}{q}} M^2 T^{\frac{2-\sigma}{2}} C, \\ & \left\| \left(\frac{\nabla \nabla}{\Delta} \cdot \mathbf{v} \right) \right\|_{L_t^q(0, T; L_x^r(\mathbb{R}^2))} \leq \left(\frac{\epsilon}{\sqrt{\mu}} \right)^{\frac{1}{q}} \left(M + M^2 T^{\frac{2-\sigma}{2}} \left(\frac{\epsilon}{\sqrt{\mu}} \right)^{\frac{\sigma}{2}} \right) C, \\ & \left\| \nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{v} - \tilde{\mathbf{v}} \right\|_{L_t^\infty(0, T; L_x^2(\mathbb{R}^2))} \leq \left(\frac{\epsilon}{\sqrt{\mu}} \right)^{\frac{2}{q}} \left(M + M^2 T^{\frac{2-\sigma}{2}} \left(\frac{\epsilon}{\sqrt{\mu}} \right)^{\frac{\sigma}{2}} \right) M T^{\frac{2-\sigma}{2}} e^{CMT} C. \end{aligned}$$

Furthermore, let $p = 2$ if $|g'| + |g''| > 0$ on \mathbb{R}^+ and $p = 3$ otherwise, there exists a constant $C > 0$ depending only on a, b, c, d such that for any $\tilde{q}, \tilde{r} \geq 2$ with $\frac{1}{\tilde{q}} + \frac{1}{p\tilde{r}} = \frac{1}{2p}$

$$\begin{aligned} & \left\| \left(\frac{\nabla \nabla}{\Delta} \cdot \mathbf{v} \right) - \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial_1}{\Delta} \\ 0 & \frac{\partial_2}{\Delta} \end{pmatrix} \exp\left(-\frac{t}{\epsilon} A(D)\right) \left(\nabla \cdot \mathbf{v}_0 \right) \right\|_{L_t^{\tilde{q}}(0, T; L_x^{\tilde{r}}(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{2p} + \frac{1}{\tilde{q}}} M^2 T^{\frac{2p-1}{2p}} C, \\ & \left\| \left(\frac{\nabla \nabla}{\Delta} \cdot \mathbf{v} \right) \right\|_{L_t^{\tilde{q}}(0, T; L_x^{\tilde{r}}(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{\tilde{q}}} \left(M + M^2 T^{\frac{2p-1}{2p}} \epsilon^{\frac{1}{2p}} \right) C, \\ & \left\| \nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{v} - \tilde{\mathbf{v}} \right\|_{L_t^\infty(0, T; L_x^2(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{2p}} \left(M + M^2 T^{\frac{2p-1}{2p}} \epsilon^{\frac{1}{2p}} \right) M T^{\frac{2p-1}{2p}} e^{CMT} C. \end{aligned}$$

Finally denoting $p_0 = 1$ if $|g'| > 0$ on \mathbb{R}^+ , $p_0 = 2$ if $|g'| + |g''| > 0$ on \mathbb{R}^+ and $p_0 = 3$ otherwise, we have

$$\sup_{x_0 \in \mathbb{R}^2} \left\| e^{-(x-x_0)^2} \left(\nabla \frac{\nabla}{\Delta} \cdot \mathbf{v} \right) \right\|_{L_t^2(0, T; L_x^2(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{2p_0}} (M + M^2 T) C.$$

If we denote by m the maximum among the multiplicities of positive zeros of g'' then one can take $\sigma = \frac{m+4}{2m+4}$. Again, the key points are dispersive estimates that can be obtained from Lemmas C2, C3, and 4.

Finally, a similar result can be obtained in the case $n = 2$ with $a + b + c + d = 0$.

Theorem 7. Let $n = 2$. Let a, b, c, d satisfying (14) with $a + b + c + d = 0$. Let $M > 0, T > 0, \epsilon \in (0, 1]$, and $\mu \in (0, 1]$. Let $(\zeta, \mathbf{v}) \in C([0, T]; (H^6 \times H^6)(\mathbb{R}^2))$ be a solution of (13) with initial datum

with $\beta = 1, s = \alpha$ if $\ell = 0$ and $\beta = 1, s = \frac{\alpha-1}{2}$ if $\ell \neq 0$.

(ζ_0, \mathbf{V}_0) , and let $\tilde{\mathbf{V}} \in C([0, T]; L^2(\mathbb{R}^2))$ be a solution of the incompressible Euler equation (9) with initial datum $\nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V}_0$ such that

$$\|(\zeta, \mathbf{V})\|_{L^\infty(0, T; (H^6 \times H^6)(\mathbb{R}^2))} + \|\tilde{\mathbf{V}}\|_{L^\infty(0, T; L^2(\mathbb{R}^2))} \leq M.$$

If a, b, c, d satisfies (15) and $g' > 0$ on \mathbb{R}^+ , there exists $\sigma \in (\frac{1}{2}, \frac{4}{5}]$ and $C > 0$ depending only on a, b, c, d , such that for any $q, r \geq 2$ with $\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}$

$$\begin{aligned} & \left\| \left(\frac{\nabla^\perp \zeta}{\Delta} \cdot \mathbf{V} \right) - \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial_1}{\Delta} \\ 0 & \frac{\partial_2}{\Delta} \end{pmatrix} \exp\left(-\frac{t}{\epsilon} A(D)\right) \begin{pmatrix} \zeta_0 \\ \nabla \cdot \mathbf{V}_0 \end{pmatrix} \right\|_{L_t^q(0, T; L_x^r(\mathbb{R}^2))} \leq \left(\frac{\epsilon}{\mu^4} \right)^{\frac{\sigma}{2} + \frac{1}{q}} M^2 T^{\frac{2-\sigma}{2}} C, \\ & \left\| \left(\frac{\nabla^\perp \zeta}{\Delta} \cdot \mathbf{V} \right) \right\|_{L_t^q(0, T; L_x^r(\mathbb{R}^2))} \leq \left(\frac{\epsilon}{\mu^4} \right)^{\frac{1}{q}} \left(M + M^2 T^{\frac{2-\sigma}{2}} \left(\frac{\epsilon}{\mu^4} \right)^{\frac{\sigma}{2}} \right) C, \\ & \left\| \nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V} - \tilde{\mathbf{V}} \right\|_{L_t^\infty(0, T; L_x^2(\mathbb{R}^2))} \leq \left(\frac{\epsilon}{\mu^4} \right)^{\frac{\sigma}{2}} \left(M + M^2 T^{\frac{2-\sigma}{2}} \left(\frac{\epsilon}{\mu^4} \right)^{\frac{\sigma}{2}} \right) M T^{\frac{2-\sigma}{2}} e^{CMT} C. \end{aligned}$$

Furthermore, let $p = 2$ if $|g'| + |g''| > 0$ on \mathbb{R}^+ and $p = 3$; otherwise, there exists a constant $C > 0$ depending only on a, b, c, d such that for any $\tilde{q}, \tilde{r} \geq 2$ with $\frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} = \frac{1}{2p}$

$$\begin{aligned} & \left\| \left(\frac{\nabla^\perp \zeta}{\Delta} \cdot \mathbf{V} \right) - \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial_1}{\Delta} \\ 0 & \frac{\partial_2}{\Delta} \end{pmatrix} \exp\left(-\frac{t}{\epsilon} A(D)\right) \begin{pmatrix} \zeta_0 \\ \nabla \cdot \mathbf{V}_0 \end{pmatrix} \right\|_{L_t^{\tilde{q}}(0, T; L_x^{\tilde{r}}(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{2p} + \frac{1}{\tilde{q}}} M^2 T^{\frac{2p-1}{2p}} C, \\ & \left\| \left(\frac{\nabla^\perp \zeta}{\Delta} \cdot \mathbf{V} \right) \right\|_{L_t^{\tilde{q}}(0, T; L_x^{\tilde{r}}(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{\tilde{q}}} \left(M + M^2 T^{\frac{2p-1}{2p}} \epsilon^{\frac{1}{2p}} \right) C, \\ & \left\| \nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V} - \tilde{\mathbf{V}} \right\|_{L_t^\infty(0, T; L_x^2(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{2p}} \left(M + M^2 T^{\frac{2p-1}{2p}} \epsilon^{\frac{1}{2p}} \right) M T^{\frac{2p-1}{2p}} e^{CMT} C. \end{aligned}$$

Finally, denoting $p_0 = 1$ if $|g'| > 0$ on \mathbb{R}^+ , $p_0 = 2$ if $|g'| + |g''| > 0$ on \mathbb{R}^+ and $p_0 = 3$ otherwise, we have

$$\sup_{x_0 \in \mathbb{R}^2} \left\| e^{-(x-x_0)^2} \left(\frac{\nabla^\perp \zeta}{\Delta} \cdot \mathbf{V} \right) \right\|_{L_t^2(0, T; L_x^2(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{2p_0}} (M + M^2 T) C.$$

If we denote by m the maximum among the multiplicities of positive zeros of g'' then one can take $\sigma = \min(\frac{m+4}{2m+4}, \frac{4}{5})$. Again the key points are dispersive estimates that can be obtained from Lemmas C2, ^{|||} C3, and 4.

5 | THE GREEN-NAGHDI EQUATIONS

The Green–Naghdi equations read as

$$\begin{cases} \epsilon \partial_t \zeta + \nabla \cdot ([1 + \epsilon \zeta] \mathbf{V}) = 0, \\ \epsilon(1 + \mu \mathcal{T}[\epsilon \zeta]) \partial_t \mathbf{V} + \nabla \zeta + \epsilon(\mathbf{V} \cdot \nabla) \mathbf{V} + \epsilon \mu \mathcal{Q}[\epsilon \zeta](\mathbf{V}) = 0 \end{cases}, \tag{16}$$

where

$$\begin{aligned} \mathcal{T}[\epsilon \zeta] \mathbf{W} &= -\frac{1}{3(1 + \epsilon \zeta)} \nabla [(1 + \epsilon \zeta)^3 \nabla \cdot \mathbf{W}] \\ \mathcal{Q}[\epsilon \zeta](\mathbf{V}) &= -\frac{1}{3(1 + \epsilon \zeta)} \nabla [(1 + \epsilon \zeta)^3 ((\mathbf{V} \cdot \nabla)(\nabla \cdot \mathbf{V}) - (\nabla \cdot \mathbf{V})^2)]. \end{aligned}$$

As before, in the 1D case if we denote by $\mathbf{U} = (\zeta, V)^T$ we have the following system:

$$\epsilon \partial_t \mathbf{U} + A(\partial_x) \mathbf{U} = \epsilon F(\zeta, V),$$

where

$$\begin{aligned} A(\partial_x) &= \begin{pmatrix} 0 & \partial_x \\ \left(1 - \frac{\mu}{3} \partial_x^2\right)^{-1} \partial_x & 0 \end{pmatrix} \\ F(\zeta, V) &= -\begin{pmatrix} \partial_x(\zeta V) \\ \left(1 - \frac{\mu}{3} \partial_x^2\right)^{-1} \left(\partial_x \left(\frac{1}{2} V^2\right) + \mu \mathcal{Q}[\epsilon \zeta](V) + \mu \mathcal{T}[\epsilon \zeta] \partial_t V + \frac{\mu}{3} \partial_x^2 \partial_t V\right) \end{pmatrix}, \end{aligned}$$

whereas in the 2D case if we denote by $\mathbf{U} = (\zeta, \nabla \cdot \mathbf{V})^T$, we get

$$\epsilon \partial_t \mathbf{U} + A(D) \mathbf{U} = \epsilon F(\zeta, \mathbf{V}),$$

where

$$\begin{aligned} A(D) &= \begin{pmatrix} 0 & 1 \\ \left(1 - \frac{\mu}{3} \Delta\right)^{-1} \Delta & 0 \end{pmatrix} \\ F(\zeta, \mathbf{V}) &= -\begin{pmatrix} \nabla \cdot (\zeta \mathbf{V}) \\ \left(1 - \frac{\mu}{3} \Delta\right)^{-1} \nabla \cdot \left((\mathbf{V} \cdot \nabla) \mathbf{V} + \mu \mathcal{Q}[\epsilon \zeta](\mathbf{V}) + \mu \mathcal{T}[\epsilon \zeta] \partial_t \mathbf{V} + \frac{\mu}{3} \Delta \partial_t \mathbf{V}\right) \end{pmatrix}. \end{aligned}$$

^{|||} with $\beta = 3, s = \alpha$ if $\ell = 0$ and $\beta = 3, s = \frac{\alpha-1}{2}$ if $\ell \neq 0$.

We refer to Refs. 23 and 32 (see also Refs. 33 and 34) for the existence of solutions of (16) on an existence time independent of ϵ that is uniform with respect to $\mu \in (0, 1]$.

We consider now the case $n = 1$.

Theorem 8. *Let $M > 0$, $T > 0$, $h_0 > 0$, $\epsilon \in (0, 1]$, and $\mu \in (0, 1]$. Let $(\zeta, V) \in C([0, T]; (H^5 \times H^5)(\mathbb{R}))$ a solution of (16) with initial datum (ζ_0, V_0) such that*

$$\|(\zeta, V)\|_{L^\infty(0,T;H^5(\mathbb{R}) \times H^5(\mathbb{R}))} \leq M \text{ and } 1 + \epsilon\zeta \geq h_0 \text{ on } [0, T].$$

There exists a constant $C_d > 0$ polynomial in M and $1/h_0$ such that for any $q, r \geq 2$ with $\frac{1}{q} + \frac{1}{2r} = \frac{1}{6}$

$$\begin{aligned} \left\| \begin{pmatrix} \zeta \\ V \end{pmatrix} - e^{-\frac{t}{\epsilon}A(\partial_x)} \begin{pmatrix} \zeta_0 \\ V_0 \end{pmatrix} \right\|_{L_t^q(0,T;L_x^r(\mathbb{R}))} &\leq \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{6}+\frac{1}{q}} T^{\frac{5}{6}} C_d, \\ \left\| \begin{pmatrix} \zeta \\ V \end{pmatrix} \right\|_{L_t^q(0,T;L_x^r(\mathbb{R}))} &\leq \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{q}} \left(1 + T^{\frac{5}{6}} \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{6}}\right) C_d, \\ \sup_{x_0 \in \mathbb{R}} \left\| e^{-(x-x_0)^2} \begin{pmatrix} \zeta \\ V \end{pmatrix} \right\|_{L_t^2(0,T;L_x^2(\mathbb{R}))} &\leq \epsilon^{\frac{1}{2}} (1 + T) C_d. \end{aligned}$$

The proof follows from the same strategy as the proof of Theorem 1 together with (6) and (7).*** One must control two new terms. Standard product estimates provide

$$\begin{aligned} \|Q[\epsilon\zeta](\mathbf{V})\|_{W_x^{2,1}} &\lesssim C\left(\frac{1}{h_0}, \|\zeta\|_{H^3}\right) \|V\|_{H^5}^2 \\ \|\mu\mathcal{T}[\epsilon\zeta]\partial_t V + \frac{\mu}{3}\partial_x^2\partial_t V\|_{W_x^{2,1}} &\lesssim C\left(\frac{1}{h_0}, \|\zeta\|_{H^3}\right) \|\epsilon\mu\partial_x\partial_t V\|_{H^3} \end{aligned}$$

and using, for instance, ideas from the proofs of Ref. [23, Lemmas 1 and 2] and standard product estimates, we obtain

$$\begin{aligned} \|\epsilon\mu\partial_x\partial_t V\|_{H^3} &\lesssim \epsilon C\left(\frac{1}{h_0}, \|\zeta\|_{H^3}\right) \|\nabla\zeta + \epsilon(V \cdot \nabla)V + \epsilon\mu Q[\epsilon\zeta](V)\|_{H^2} \\ &\lesssim C\left(\frac{1}{h_0}, \|\zeta\|_{H^3}, \|V\|_{H^5}\right). \end{aligned}$$

Remark 7. As noted in Remark 2 concerning Theorem 1, in the case $\mu = \mathcal{O}(\epsilon)$ the first estimate of Theorem 8 does not provide a convergence result as $\epsilon \rightarrow 0$ so that nonlinear terms must be taken into account and asymptotic models like a system of decoupling KdV equations or decoupling BBM equations become relevant. We refer to Ref. [21, Chapter 7].

*** Note that the source term F is not a derivative here so that one can not use (5).

We now consider the case $n = 2$. Applying the operator $\nabla^\perp \cdot$ to the second equation of (16) and denoting by $\omega := \nabla^\perp \cdot \mathbf{V}$, we get the following equation:

$$\partial_t \omega + (\mathbf{V} \cdot \nabla)\omega + (\nabla \cdot \mathbf{V})\omega + \frac{\epsilon \mu \nabla \zeta^\perp}{3(1 + \epsilon \zeta)^2} \cdot \nabla [(1 + \epsilon \zeta)^3 (\nabla \cdot \partial_t \mathbf{V} + (\mathbf{V} \cdot \nabla)(\nabla \cdot \mathbf{V}) - (\nabla \cdot \mathbf{V})^2)] = 0.$$

Theorem 9. *Let $M > 0$, $T > 0$, $h_0 > 0$, $\epsilon \in (0, 1]$, and $\mu \in (0, 1]$. Let $(\zeta, \mathbf{V}) \in C([0, T]; (H^9 \times H^9)(\mathbb{R}^2))$ be a solution of (16) with initial datum (ζ_0, \mathbf{V}_0) and $\tilde{\mathbf{V}} \in C([0, T]; L^2(\mathbb{R}^2))$ be a solution of the incompressible Euler equation (9) with initial datum $\nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V}_0$ such that*

$$\|(\zeta, \mathbf{V})\|_{L^\infty(0,T;(H^9 \times H^9)(\mathbb{R}^2))} + \|\tilde{\mathbf{V}}\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} \leq M \text{ and } 1 + \epsilon \zeta \geq h_0 \text{ on } [0, T].$$

There exists a constant $C_d > 0$ polynomial in M and $1/h_0$ and a universal constant $C > 0$ such that for any $q, r \geq 2$ with $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$

$$\left\| \left(\frac{\nabla \nabla}{\Delta} \cdot \zeta, \mathbf{V} \right) - \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial_1}{\Delta} \\ 0 & \frac{\partial_2}{\Delta} \end{pmatrix} \exp\left(-\frac{t}{\epsilon} A(D)\right) \begin{pmatrix} \zeta_0 \\ \nabla \cdot \mathbf{V}_0 \end{pmatrix} \right\|_{L_t^q(0,T;L_x^r(\mathbb{R}^2))} \leq \left(\frac{\epsilon}{\sqrt{\mu}} \ln\left(1 + \frac{\mu}{\epsilon^2} T\right) \right)^{\frac{1}{2} + \frac{1}{q}} T^{\frac{1}{2}} C_d + \frac{\epsilon}{\sqrt{\mu}} T C_d,$$

$$\left\| \left(\frac{\nabla \nabla}{\Delta} \cdot \zeta, \mathbf{V} \right) \right\|_{L_t^q(0,T;L_x^r(\mathbb{R}^2))} \leq \left(\frac{\epsilon}{\sqrt{\mu}} \ln\left(1 + \frac{\mu}{\epsilon^2} T\right) \right)^{\frac{1}{q}} \left(1 + \left(\frac{\epsilon}{\sqrt{\mu}} \ln\left(1 + \frac{\mu}{\epsilon^2} T\right) T \right)^{\frac{1}{2}} \right) C_d + \frac{\epsilon}{\sqrt{\mu}} (1 + T) C_d,$$

$$\left\| \nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{V} - \tilde{\mathbf{V}} \right\|_{L_t^\infty(0,T;L_x^2(\mathbb{R}^2))} \leq \left(\frac{\epsilon}{\sqrt{\mu}} \ln\left(1 + \frac{\mu}{\epsilon^2} T\right) \right)^{\frac{1}{2}} \left(1 + \left(\frac{\epsilon}{\sqrt{\mu}} \ln\left(1 + \frac{\mu}{\epsilon^2} T\right) T \right)^{\frac{1}{2}} \right) \sqrt{T} e^{CMT} C_d + \frac{\epsilon}{\sqrt{\mu}} (1 + T) \sqrt{T} e^{CMT} C_d,$$

and there exists a constant $\tilde{C}_d > 0$ polynomial in M and $1/h_0$ and a universal constant $\tilde{C} > 0$ such that for any $\tilde{q}, \tilde{r} \geq 2$ with $\frac{1}{\tilde{q}} + \frac{1}{2\tilde{r}} = \frac{1}{4}$

$$\left\| \left(\frac{\nabla \nabla}{\Delta} \cdot \zeta, \mathbf{V} \right) - \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial_1}{\Delta} \\ 0 & \frac{\partial_2}{\Delta} \end{pmatrix} \exp\left(-\frac{t}{\epsilon} A(D)\right) \begin{pmatrix} \zeta_0 \\ \nabla \cdot \mathbf{V}_0 \end{pmatrix} \right\|_{L_t^{\tilde{q}}(0,T;L_x^{\tilde{r}}(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{4} + \frac{1}{\tilde{q}}} T^{\frac{3}{4}} \tilde{C}_d,$$

$$\left\| \left(\frac{\nabla \nabla}{\Delta} \cdot \zeta, \mathbf{V} \right) \right\|_{L_t^{\tilde{q}}(0,T;L_x^{\tilde{r}}(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{\tilde{q}}} \left(1 + \epsilon^{\frac{1}{4}} T^{\frac{3}{4}} \right) \tilde{C}_d,$$

$$\left\| \nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot \mathbf{v} - \tilde{\mathbf{v}} \right\|_{L_t^\infty(0,T;L_x^2(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{4}} \left(1 + \epsilon^{\frac{1}{4}} T^{\frac{3}{4}} \right) T^{\frac{3}{4}} e^{\tilde{C}MT} \tilde{C}_d,$$

$$\sup_{x_0 \in \mathbb{R}^2} \left\| e^{-(x-x_0)^2} \left(\nabla \frac{\zeta}{\Delta} \cdot \mathbf{v} \right) \right\|_{L_t^2(0,T;L_x^2(\mathbb{R}^2))} \leq \epsilon^{\frac{1}{2}} (1 + T) \tilde{C}_d.$$

The proof follows from the same strategy as the proofs of Theorems 2 and 3. One must control two new terms. Standard product estimates provide

$$\| \mathcal{Q}[\epsilon\zeta](\mathbf{v}) \|_{W_x^{4,1}} \lesssim C \left(\frac{1}{h_0}, \|\zeta\|_{H^5} \right) \|\mathbf{v}\|_{H^7}^2$$

$$\left\| -\frac{\mu}{3} ((1 + \epsilon\zeta)^2 - 1) \nabla \nabla \cdot \partial_t \mathbf{v} - \frac{\epsilon\mu}{3} (1 + \epsilon\zeta)^2 \nabla \zeta \nabla \cdot \partial_t \mathbf{v} \right\|_{W_x^{4,1}} \lesssim C \left(\frac{1}{h_0}, \|\zeta\|_{H^5} \right) \|\epsilon\mu \partial_t \nabla \cdot \mathbf{v}\|_{H^6}$$

and using, for instance, Ref. [32, Lemmas 2.1 and 2.4] and standard product estimates, we obtain

$$\begin{aligned} \|\epsilon\mu \partial_t \nabla \cdot \mathbf{v}\|_{H^6} &\lesssim C \left(\frac{1}{h_0}, \|\zeta\|_{H^6} \right) \|\nabla \zeta + \epsilon(\mathbf{v} \cdot \nabla) \mathbf{v} + \epsilon\mu \mathcal{Q}[\epsilon\zeta](\mathbf{v})\|_{H^5} \\ &\lesssim C \left(\frac{1}{h_0}, \|\zeta\|_{H^6}, \|\mathbf{v}\|_{H^8} \right). \end{aligned}$$

We note the strategy used in the proof of Theorems 2 and 3 also provides bounds on $\|\nabla \zeta\|_{L_t^2(0,T;L_x^\infty)}$. Second, a new term also appears in the control of the rotational component. We note that

$$\nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot (\mu \mathcal{T}[\epsilon\zeta] \partial_t \mathbf{v} + \mu \mathcal{Q}[\epsilon\zeta](\mathbf{v})) = -\epsilon\mu \nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot (h[\nabla \cdot \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)(\nabla \cdot \mathbf{v}) - (\nabla \cdot \mathbf{v})^2] \nabla \zeta)$$

and using previous bounds we get

$$\left\| \nabla^\perp \frac{\nabla^\perp}{\Delta} \cdot (\mu \mathcal{T}[\epsilon\zeta] \partial_t \mathbf{v} + \mu \mathcal{Q}[\epsilon\zeta](\mathbf{v})) \right\|_{L_x^2} \leq C \left(\frac{1}{h_0}, \|\zeta\|_{H^2}, \|\mathbf{v}\|_{H^3} \right) \|\nabla \zeta\|_{L_x^\infty}$$

so that the strategy used in the proof of Theorems 2 and 3 to control the vorticity component can easily be adapted.

ACKNOWLEDGMENTS

We deeply thank the anonymous referee for careful reading and very valuable comments on the manuscript. This work has been partially funded by the ANR project CRISIS (ANR-20-CE40-0020-01).

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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How to cite this article: Melinand B. Rigid lid limit in shallow water over a flat bottom. *Stud Appl Math*. 2024;153:e12773. <https://doi.org/10.1111/sapm.12773>

APPENDIX A: LITTLEWOOD–PALEY DECOMPOSITION

In this section, we introduce homogeneous and inhomogeneous Littlewood–Paley decompositions and provide basic properties. Let φ_0 be a smooth nonnegative even function supported in $[-1, 1]$, that is equal to 1 in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and that is nonincreasing on \mathbb{R}^+ . Then we define, for any $y \in \mathbb{R}$ and any $j \in \mathbb{Z}$ the function $P_j(y) := \varphi_0(2^{-j-1}y) - \varphi_0(2^{-j}y)$. We note that P_j is a function supported in the annulus $C(2^{j-1}, 2^{j+1})$ for any $j \in \mathbb{Z}$.

For any $y \in \mathbb{R}$

$$P_j(y) \in [0, 1], \varphi_0(y) + \sum_{j \in \mathbb{N}} P_j(y) = 1, \frac{1}{2} \leq \varphi_0(y)^2 + \sum_{j \in \mathbb{N}} P_j^2(y) \leq 1.$$

Then for any $p \in [1, \infty]$ and any Schwartz class function f

$$\varphi_0(|D|)f + \sum_{j=0}^N P_j(|D|)f \xrightarrow[N \rightarrow \infty]{L^p} f$$

since

$$\|(1 - \varphi_0(2^{-N-1}|D|))f\|_{L^p} \xrightarrow[N \rightarrow \infty]{} 0.$$

Such decomposition of the function f is called inhomogeneous Littlewood–Paley decomposition.

For any $y \in \mathbb{R}^*$,

$$P_j(y) \in [0, 1], \sum_{j \in \mathbb{Z}} P_j(y) = 1, \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} P_j^2(y) \leq 1.$$

Then for any $p \in (1, \infty]$ and any Schwartz class function f , we have

$$\sum_{|j| \leq N} P_j(|D|)f \xrightarrow[N \rightarrow \infty]{L^p} f$$

since

$$\|\varphi_0(2^N |D|)f\|_{L^p} \xrightarrow[N \rightarrow \infty]{} 0 \text{ and } \|(1 - \varphi_0(2^{-N-1} |D|))f\|_{L^p} \xrightarrow[N \rightarrow \infty]{} 0.$$

Such decomposition of the function f is called homogeneous Littlewood–Paley decomposition.

APPENDIX B: FOURIER MULTIPLIERS ON LEBESGUE SPACES

In this section, we gather useful estimates concerning Fourier multipliers on L^p . The first lemma is about Bessel’s potential.

Lemma B1. *Let $n = 1$ or 2 . For any $\alpha \geq 0$, there exists a constant $C > 0$ such that for any $a \geq 0$, any $p \in [1, \infty]$ and any $f \in L^p(\mathbb{R}^n)$*

$$\begin{aligned} \left\| (1 + a|D|^2)^{-\frac{\alpha}{2}} f \right\|_{L^p(\mathbb{R}^n)} &\leq C \|f\|_{L^p(\mathbb{R}^n)}, \\ \left\| a^{\frac{\alpha}{2}} |D|^\alpha (1 + a|D|^2)^{-\frac{\alpha}{2}} f \right\|_{L^p(\mathbb{R}^n)} &\leq C \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Furthermore, for any $\alpha \geq 0$ and any $b > 0$, there exists a constant $C > 0$ such that for any $a \geq 0$, any $p \in [1, \infty]$ and any $f \in L^p(\mathbb{R}^n)$

$$\left\| (1 + ba|D|^2)^{\frac{\alpha}{2}} (1 + a|D|^2)^{-\frac{\alpha}{2}} f \right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$

Finally, for any $p \in (1, \infty)$, there exists a constant $C > 0$ such that for any $f \in W^{1,p}(\mathbb{R}^n)$

$$\| |D|f \|_{L^p} \leq C \|\nabla f\|_{L^p}.$$

Proof. By homogeneity, one can assume $a = 1$. As noted in Ref. [35, V.3.1], $\mathcal{F}^{-1}((1 + 4\pi^2|\xi|^2)^{-\frac{\alpha}{2}})$ is in $L^1(\mathbb{R}^n)$ so that the first bound follows Young’s convolution inequality. The second bound is proved in Ref. [35, V.3.2].

Concerning the third point, we note from Ref. [35, V.3.2] that there exist two finite measures ν and μ on \mathbb{R}^n such that

$$(1 + ba|D|^2)^{\frac{\alpha}{2}} (1 + a|D|^2)^{-\frac{\alpha}{2}} f = \nu * (1 + a|D|^2)^{-\frac{\alpha}{2}} f + \mu * (ba)^{\frac{\alpha}{2}} |D|^\alpha (1 + a|D|^2)^{-\frac{\alpha}{2}} f$$

so that the result follows from the first point.

Finally, since $|D| = -\sum_{j=1}^n \frac{\partial_j}{|D|} \partial_j$, the last point follows from the fact that the Riesz transforms are bounded on L^p for $p \in (1, \infty)$. □

We then recall Bernstein’s lemma.

Lemma B2. *Let $n \in \mathbb{N}^*$ and $b > a > 0$. Let ϕ a smooth function supported in $[a, b]$ and χ a smooth function compactly supported. Then, for any $s \in \mathbb{R}$ and any $k \in \mathbb{N}$, there exists a constant $C > 0$ such*

that for any $\lambda > 0$, any $p, q \in [1, \infty]$ with $q \geq p$ and any $f \in L^p(\mathbb{R}^n)$

$$\begin{aligned} \|\nabla^k \chi(\lambda^{-1}|D|)f\|_{L^q(\mathbb{R}^n)} &\leq C\lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|\chi(\lambda^{-1}|D|)f\|_{L^p(\mathbb{R}^n)}, \\ \frac{1}{C}\lambda^s \|\phi(\lambda^{-1}|D|)f\|_{L^p(\mathbb{R}^n)} &\leq \||D|^s \phi(\lambda^{-1}|D|)f\|_{L^p(\mathbb{R}^n)} \leq C\lambda^s \|\phi(\lambda^{-1}|D|)f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Then, we provide a high-frequency result.

Lemma B3. *Let $\beta > 0$, $n \in \mathbb{N}^*$. Let χ be a smooth compactly supported function that is equal to 1 near 0. There exists a constant $C > 0$ such that for any $p \in [1, \infty]$, any Schwartz class function f , and any $\lambda > 0$*

$$\left\| \frac{1 - \chi(\lambda|D|)}{(\lambda|D|)^\beta} f \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \frac{1 - \chi(\lambda|D|)}{(1 + \lambda^2|D|^2)^{\frac{\beta}{2}}} f \right\|_{L^p(\mathbb{R}^n)}.$$

Proof. By homogeneity, one can assume $\lambda = 1$. Using Ref. [35, V.3.2], there exist two finite measures ν and μ on \mathbb{R}^n such that

$$\frac{(1 + |D|^2)^{\frac{\beta}{2}}}{(1 + |D|^2)^{\frac{\beta}{2}}} \frac{1 - \chi(|D|)}{|D|^\beta} f = \nu * \frac{1 - \chi(|D|)}{|D|^\beta} \frac{1}{(1 + |D|^2)^{\frac{\beta}{2}}} f + \mu * \frac{1 - \chi(|D|)}{(1 + |D|^2)^{\frac{\beta}{2}}} f.$$

Then, we get

$$\left\| \mu * \frac{1 - \chi(|D|)}{(1 + |D|^2)^{\frac{\beta}{2}}} f \right\|_{L^p} \lesssim \left\| \frac{1 - \chi(|D|)}{(1 + |D|^2)^{\frac{\beta}{2}}} f \right\|_{L^p}.$$

Furthermore, using a Littlewood–Paley decomposition as in Section A together with Bernstein’s Lemma B2, there exists an integer $k_0 \in \mathbb{Z}$ such that for any Schwartz class function g

$$\begin{aligned} \left\| \frac{1 - \chi(|D|)}{|D|^\beta} g \right\|_{L^p} &= \left\| \sum_{j \geq k_0} \frac{1 - \chi(|D|)}{|D|^\beta} P_j(|D|)g \right\|_{L^p} \leq \sum_{j \geq k_0} \left\| \frac{1 - \chi(|D|)}{|D|^\beta} P_0(2^{-j}|D|)g \right\|_{L^p} \\ &\lesssim \sum_{j \geq k_0} 2^{-\beta j} \|P_0(|D|)(1 - \chi(|D|))g\|_{L^p} \lesssim \|(1 - \chi(|D|))g\|_{L^p} \end{aligned}$$

so that

$$\left\| \nu * \frac{1 - \chi(|D|)}{|D|^\beta} \frac{1}{(1 + |D|^2)^{\frac{\beta}{2}}} f \right\|_{L^p} \lesssim \left\| \frac{1 - \chi(|D|)}{|D|^\beta} \frac{1}{(1 + |D|^2)^{\frac{\beta}{2}}} f \right\|_{L^p} \lesssim \left\| \frac{1 - \chi(|D|)}{(1 + |D|^2)^{\frac{\beta}{2}}} f \right\|_{L^p}.$$

□

In the following, we provide a boundedness result in L^1 when $n = 1$.

Lemma B4. *Let $s \in [0, 1)$. There exists a constant $C_s > 0$ such that for any $f \in W^{1,1}(\mathbb{R})$*

$$\| |D|^s f \|_{L^1(\mathbb{R})} \leq C_s \|f\|_{W^{1,1}(\mathbb{R})}$$

and there exists a constant $C > 0$ such that for any $f \in W^{2,1}(\mathbb{R})$

$$\| |D|f \|_{L^1(\mathbb{R})} \leq C \|f\|_{W^{2,1}(\mathbb{R})}.$$

Proof. Using an inhomogeneous Littlewood–Paley decomposition as in Section A, we have

$$\| |D|^s f \|_{L^1(\mathbb{R})} \leq \| \varphi_0(|D|) |D|^s f \|_{L^1(\mathbb{R})} + \sum_{j \in \mathbb{N}} \| \text{sgn}(D) |D|^{s-1} P_j(|D|) \partial_x f \|_{L^1(\mathbb{R})}.$$

Using Lemma B1, Young’s convolution inequality and since $\xi \mapsto \varphi_0(|\xi|)(1 + |\xi|^2)^{\frac{s}{2}}$ is a smooth compactly supported function

$$\| \varphi_0(|D|) |D|^s f \|_{L^1(\mathbb{R})} \lesssim \| \varphi_0(|D|)(1 + |D|^2)^{\frac{s}{2}} f \|_{L^1(\mathbb{R})} \lesssim \|f\|_{L^1(\mathbb{R})}.$$

Then we note that for any $\alpha \in \mathbb{R}$, the map $\xi \mapsto \text{sgn}(\xi)|\xi|^\alpha P_0(|\xi|)$ is a smooth compactly supported function so that

$$\| \mathcal{F}^{-1}(\text{sgn}(\xi)^k |\xi|^\alpha P_j(|\xi|)) \|_{L^1} = 2^{\alpha j} \| \mathcal{F}^{-1}(\text{sgn}(\xi)^k |\xi|^\alpha P_0(|\xi|)) \|_{L^1} \lesssim 2^{\alpha j}.$$

Therefore, it follows from Young’s convolution inequality

$$\| |D|^s f \|_{L^1(\mathbb{R})} \lesssim \|f\|_{L^1(\mathbb{R})} + \sum_{j \in \mathbb{N}} 2^{(s-1)j} \| \partial_x f \|_{L^1(\mathbb{R})},$$

and the first point follows. The second point follows the same way. □

A similar result can be obtained when $n = 2$.

Lemma B5. *Let $s \in [0, 2]$. There exists a constant $C > 0$ such that for any $f \in W^{2,1}(\mathbb{R}^2)$*

$$\| |D|^s f \|_{L^1(\mathbb{R}^2)} \leq C \|f\|_{W^{2,1}(\mathbb{R}^2)}.$$

APPENDIX C: DISPERSIVE ESTIMATES

In this section, we gather different dispersive estimates that are useful through this work. These are obtained from Ref. 31. We begin with the case $n = 1$.

Lemma C1. *Let $n = 1$. Let $\lambda > 0$, $\alpha \in \mathbb{R}$ with $\alpha \notin \{-2, -1\}$, $\beta \geq 0$, and $l \in \mathbb{N}$ with $l \geq 2$. Assume that g is an odd C^2 function. Let $y_1 > y_0 > 0$. Let χ be a smooth even compactly supported function whose support is a subset of $[-y_0, y_0]$ and that is equal to 1 on $[-\frac{1}{2}y_0, \frac{1}{2}y_0]$.*

- (i) *Let $s \in [0, \frac{\beta}{2}]$. Assume that $|g''| \geq \lambda y^\beta$ on $[0, y_0]$ and, if $s = \frac{\beta}{2}$, that $|g' - g(0)| \geq \lambda y^{\beta+1}$ on $[0, y_0]$. There exists $C > 0$ such that for any $\mu > 0$, any $t \in \mathbb{R}^*$, any $m \in \{0, 1\}$, and any Schwartz*

class function f

$$\left\| e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}D)} \chi(\sqrt{\mu}D) (\text{sgn}(D))^m |D|^s f \right\|_{L_x^\infty} \leq \frac{C}{|t|^{\frac{s+1}{2+\beta}}} \mu^{-\frac{(\beta+1)(s+1)}{2(2+\beta)}} \left\| \chi(\sqrt{\mu}|D|) f \right\|_{L^1}.$$

(ii) Assume that g is $C^l(\mathbb{R})$. Assume that $\sum_{p=2}^l |g^{(p)}| \geq \lambda$ on $[\frac{1}{2}y_0, 2y_1]$, that $|g''| \geq \lambda y^\alpha$ on $[y_1, \infty)$ and, if $l = 2$, that $\frac{1}{\lambda} y^{\alpha+1} \geq |g' - a| \geq \lambda y^{\alpha+1}$ on $[y_1, \infty)$ for some $a \in \mathbb{R}$. There exists $C > 0$ such that for any $\mu > 0$, any $t \in \mathbb{R}^*$ and any Schwartz class function f

$$\left\| e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}D)} (1 - \chi(\sqrt{\mu}D)) f \right\|_{L_x^\infty} \leq \frac{C}{|t|^{\frac{1-l}{i}}} \mu^{\frac{1-l}{2i}} \left\| (\sqrt{\mu}|D|)^{\frac{l-2-\alpha}{i}} (1 - \chi(\sqrt{\mu}D)) f \right\|_{L^1}.$$

Proof. We introduce a homogeneous Littlewood–Paley decomposition as in Section A. There exists $k_0 \in \mathbb{Z}$ such that using Young’s convolution inequality and Bernstein’s Lemma B2

$$\begin{aligned} & \left\| e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}D)} \chi(\sqrt{\mu}D) (\text{sgn}(D))^m |D|^s f \right\|_{L_x^\infty} \\ &= \left\| \sum_{k \leq k_0} e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}D)} P_k(\sqrt{\mu}D) (\text{sgn}(D))^m |D|^s \chi(\sqrt{\mu}|D|) f \right\|_{L_x^\infty} \\ &\lesssim \left\| \sum_{k \leq k_0} \mathcal{F}^{-1} \left(e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}D)} P_k(\sqrt{\mu}D) (\text{sgn}(D))^m |D|^s \right) \right\|_{L_x^\infty} \left\| \chi(\sqrt{\mu}|D|) f \right\|_{L^1}. \end{aligned}$$

The first inequality follows from Ref. [31, Lemma 2.6].

Second, if \hat{f} is compactly supported there exists $k_2, k_1 \in \mathbb{Z}$ such that using Young’s convolution inequality

$$\begin{aligned} & \left\| e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}D)} (1 - \chi(\sqrt{\mu}D)) f \right\|_{L_x^\infty} = \left\| \sum_{k_2 \geq k \geq k_1} e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}D)} P_k(\sqrt{\mu}D) (1 - \chi(\sqrt{\mu}D)) f \right\|_{L_x^\infty} \\ &\lesssim \left\| \sum_{k_2 \geq k \geq k_1} \mathcal{F}^{-1} \left(e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}D)} P_k(\sqrt{\mu}D) |D|^s \right) \right\|_{L_x^\infty} \left\| |D|^{-s} (1 - \chi(\sqrt{\mu}D)) f \right\|_{L^1} \end{aligned}$$

with $s = -\frac{l-2-\alpha}{i}$. The second inequality follows from Ref. [31, Lemmas 2.6 and 2.9] and by density of $\mathcal{F}^{-1}(C_c^\infty(\mathbb{R}))$. □

Then, we consider the case $n = 2$.

Lemma C2. Let $n = 2$. Let $\lambda > 0$, $m \in \mathbb{N}$, $\beta \geq 1$, and $\alpha \in \mathbb{R}$ with $\alpha \notin \{-2, -1\}$. Assume that g is $C^3(\mathbb{R})$. Let $y_0 > 0$. Let χ be a smooth even compactly supported function whose support is a subset of $[-y_0, y_0]$ and that is equal to 1 on $[0, \frac{1}{2}y_0]$.

(i) Assume that $|g''| \geq \lambda y^\beta$, $|g' - g'(0)| \geq \lambda y^{\beta+1}$, and $|g'| \geq \lambda$ on $[0, y_0]$. There exists $C > 0$ such that for any $\mu > 0$, any $t \in \mathbb{R}^*$, and any Schwartz class function f

$$\left\| e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}|D|)} \chi(\sqrt{\mu}|D|) \left(\frac{\nabla}{|D|} \right)^m f \right\|_{L^\infty} \leq \frac{C}{\mu} \left(\frac{\sqrt{\mu}}{|t|} \right)^{\frac{5+\beta}{2(2+\beta)}} \left\| \chi(\sqrt{\mu}|D|) f \right\|_{L^1}.$$

(ii) Let $s \in \mathbb{R}$ such that $(s + 2)(s - \alpha) < 0$ or $s = \alpha$. Assume that $|g'| \geq \lambda y^{\alpha+1}$ and $|g''| \geq \lambda y^\alpha$ on $[\frac{y_0}{2}, \infty)$ and, if $s = \alpha$, that $|g'| \leq \frac{1}{\lambda} y^{\alpha+1}$, $|g''| \leq \frac{1}{\lambda} y^\alpha$, and $|g'''| \leq \frac{1}{\lambda} y^{\alpha-1}$ on $[\frac{y_0}{2}, \infty)$. There exists $C > 0$ such that for any $\mu > 0$, any $t \in \mathbb{R}^*$, and any Schwartz class function f

$$\left\| e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}|D|)} (1 - \chi(\sqrt{\mu}|D|)) \left(\frac{\nabla}{|D|} \right)^m f \right\|_{L^\infty} \leq \frac{C}{\mu} \left(\frac{\sqrt{\mu}}{|t|} \right)^{\frac{s+2}{2+\alpha}} \left\| (\sqrt{\mu}|D|)^{-s} (1 - \chi(\sqrt{\mu}|D|)) f \right\|_{L^1}.$$

(iii) Assume that $\alpha < -1$. Let $s \in \mathbb{R}$ such that $(s + 2)(s - \frac{\alpha-1}{2}) < 0$ or $s = \frac{\alpha-1}{2}$. Assume that $|g'| \geq \lambda$ and $|g''| \geq \lambda y^\alpha$ on $[\frac{y_0}{2}, \infty)$ and, if $s = \frac{\alpha-1}{2}$, that $\frac{1}{\lambda} y^{\alpha+1} \geq |g' - a| \geq \lambda y^{\alpha+1}$ on $[\frac{y_0}{2}, \infty)$ for some $a \in \mathbb{R}^*$. There exists $C > 0$ such that for any $\mu > 0$, any $t \in \mathbb{R}^*$, and any Schwartz class function f

$$\left\| e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}|D|)} (1 - \chi(\sqrt{\mu}|D|)) \left(\frac{\nabla}{|D|} \right)^m f \right\|_{L^\infty} \leq \frac{C}{\mu} \left(\frac{\sqrt{\mu}}{|t|} \right)^{\frac{2(s+2)}{3+\alpha}} \left\| (\sqrt{\mu}|D|)^{-s} (1 - \chi(\sqrt{\mu}|D|)) f \right\|_{L^1}.$$

Proof. We begin with the case $m = 0$. Point (i) follows directly from Ref. [31, Lemma 2.12]. Concerning points (ii) and (iii), by introducing a Littlewood–Paley decomposition as in Section A and proceeding as the previous lemma for high frequencies, Points (ii) and (iii) follow, respectively, from Lemma Ref. [31, Lemma 2.15] and Ref. [31, Lemma 2.17].

We now consider the case $m \geq 1$. We claim that one can easily adapt the estimates in Ref. [31, Section 2.4] to our setting. Indeed, in our case, one must estimate integrals under the form $\int_{\mathbb{R}^+} e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}r)} \tilde{J}(r|x|) \chi(\sqrt{\mu}r) r dr$ or $\int_{\mathbb{R}^+} e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}r)} \tilde{J}(r|x|) P(\frac{\sqrt{\mu}r}{2^k}) r^{s+1} dr$, where $\tilde{J}(\tau) = \int_0^{2\pi} e^{i\tau \sin(\theta)} u(\theta) d\theta$ and u is a smooth periodic function ($u \equiv 1$ in Ref. [31, Section 2.4]). Similarly as $J_0(\tau) := \int_0^{2\pi} e^{i\tau \sin(\theta)} d\theta$, one can decompose \tilde{J} as $\tilde{J}(\tau) = \tilde{h}_-(\tau) e^{i\tau} + \tilde{h}_+(\tau) e^{-i\tau}$ where, for any $p \in \mathbb{N}$, $|\tilde{h}_\pm^{(p)}(\tau)| \lesssim (1 + |\tau|)^{-p-\frac{1}{2}}$. Then, one can adapt all the results of Ref. [31, Section 2.4] replacing J_0 by \tilde{J} so that the strategy used to prove the case $m = 0$ also works. \square

Lemma C3. Let $n = 2$. Let $\lambda > 0$, $m \in \mathbb{N}$. Assume that g is $C^2(\mathbb{R})$. Let $y_1 > y_0 > 0$. Let χ be a smooth even compactly supported function whose support is a subset of $[-y_0, y_0]$ and that is equal to 1 on $[0, \frac{1}{2}y_0]$, ϕ a smooth function supported in $[\frac{1}{2}, 2]$ and $\tilde{\chi}$ a smooth compactly supported function whose support is a subset of $[y_0, y_1]$

(i) Assume that $|g'| \geq \lambda$ and g'' has a finite number of zeros on $(0, y_0]$. There exists $C > 0$ such that for any $\mu > 0$, any $t \in \mathbb{R}^*$, any $l \in \mathbb{N}$, with $l \geq 2$, any $k \in \mathbb{Z}$ and any Schwartz class function f

$$\left\| e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}|D|)} \chi(\sqrt{\mu}|D|) \phi(2^{-k} \sqrt{\mu}|D|) \left(\frac{\nabla}{|D|} \right)^m f \right\|_{L^\infty} \leq \frac{C}{|t|^{\frac{1}{l}}} \left(\frac{2^k}{\sqrt{\mu}} \right)^{2-\frac{1}{l}} \left\| \phi(2^{-k} \sqrt{\mu}|D|) f \right\|_{L^1}.$$

In particular, for any $\eta > 0$, there exists $C_\eta > 0$ such that for any $\mu > 0$, any $t \in \mathbb{R}^*$, any $l \in \mathbb{N}$ with $l \geq 2$, and any Schwartz class function f

$$\left\| e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}|D|)} \chi(\sqrt{\mu}|D|) \left(\frac{\nabla}{|D|} \right)^m f \right\|_{L_x^\infty} \leq \frac{C_\eta}{|t|^{\frac{1}{l}}} \left(\| |D|^{2-\frac{1}{l}-\eta} f \|_{L^1} + \| |D|^{2-\frac{1}{l}+\eta} f \|_{L^1} \right).$$

(ii) Assume that g is $C^l(\mathbb{R})$, that $\sum_{p=1}^l |g^{(p)}| \geq \lambda$ and g'' has a finite number of zeros on $[y_0, y_1]$. There exists $C > 0$ such that for any $t \in \mathbb{R}^*$ and any $\mu > 0$

$$\left\| e^{i \frac{t}{\sqrt{\mu}} g(\sqrt{\mu}|D|)} \tilde{\chi}(\sqrt{\mu}|D|) \left(\frac{\nabla}{|D|} \right)^m f \right\|_{L_x^\infty} \leq \frac{C}{|t|^{\frac{1}{l}}} \| |D|^{2-\frac{1}{l}} f \|_{L^1}.$$

Proof. The first inequality is an easy adaptation of Ref. [31, Lemma 2.21(1)]. The second bound is a consequence of the first bound together with the use of a Littlewood–Paley decomposition, Bernstein’s Lemma B2 and the fact that

$$\sum_{2^k \leq \sqrt{\mu}} \left(\frac{2^k}{\sqrt{\mu}} \right)^\eta \lesssim_\eta 1, \quad \sum_{\sqrt{\mu} \leq 2^k \leq 2y_0} \left(\frac{2^k}{\sqrt{\mu}} \right)^{-\eta} \leq \sum_{\sqrt{\mu} \leq 2^k} \left(\frac{2^k}{\sqrt{\mu}} \right)^{-\eta} \lesssim_\eta 1.$$

The third inequality easily follows from Ref. [31, Lemma 2.21(2)]. □

Finally, we provide Morawetz-type estimates.

Proposition 4. Let $n \in \mathbb{N}^*$ and $T > 0$. Assume that g is $C^1(\mathbb{R}_+^*)$. There exists $C > 0$ independent of T such that for any function $f \in L^2(\mathbb{R}^n)$, any function F in $L^\infty(0, T; L^2(\mathbb{R}^n))$, any $\mu > 0$, any $\epsilon > 0$, any $a > 0$, and any $x_0 \in \mathbb{R}^n$

$$\int_0^T \int_{\mathbb{R}^n} \left| |g'(\sqrt{\mu}|D|)|^{\frac{1}{2}} e^{i \frac{t}{\epsilon\sqrt{\mu}} g(\sqrt{\mu}|D|)} f \right|^2 e^{-\frac{a}{2}|x-x_0|^2} dx dt \leq \epsilon \frac{C}{\sqrt{a}} \|f\|_{L_x^2}^2,$$

and

$$\int_0^T \int_{\mathbb{R}^n} \left| \int_0^t |g'(\sqrt{\mu}|D|)|^{\frac{1}{2}} e^{i \frac{(t-s)}{\epsilon\sqrt{\mu}} g(\sqrt{\mu}|D|)} F(s, \cdot) ds \right|^2 e^{-\frac{a}{2}|x-x_0|^2} dx dt \leq \epsilon T^2 \frac{C}{\sqrt{a}} \|F\|_{L_s^2(0,T;L_x^2)}^2.$$

Proof. After an appropriate change of variable in time, we get from Ref. [31, Proposition 2.28]

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \left| (|g'(\sqrt{\mu}|D|)|^{\frac{1}{2}} e^{i \frac{t}{\epsilon\sqrt{\mu}} g(\sqrt{\mu}|D|)} f)(x) \right|^2 e^{-\frac{a}{2}|x-x_0|^2} dx dt \leq \epsilon \frac{C}{\sqrt{a}} \|f\|_{L^2}^2.$$

Then denoting by I the second quantity to bound and using Jensen’s inequality and the previous estimate we obtain

$$I \leq \int_0^T T \left\| (|g'(\sqrt{\mu}|D|)|^{\frac{1}{2}} e^{i \frac{(t-s)}{\epsilon\sqrt{\mu}} g(\sqrt{\mu}|D|)} F(s, \cdot)) e^{-\frac{a}{4}|x-x_0|^2} \right\|_{L_t^2(s,T;L_x^2)}^2 ds \leq \epsilon T^2 \frac{C}{\sqrt{a}} \|F\|_{L_s^2(0,T;L_x^2)}^2.$$

□